DESIGN TECHNIQUES FOR A CLASS OF MULTIRATE SAMPLED DATA CONTROL SYSTEMS

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DESIGN TECHNIQUES FOR A CLASS OF

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SUMMARY

The objectives of this research are: (1) to develop a general procedure for frequency domain design of a class of linear time invariant multirate multiloop sampled data control systems, and (2) to modify two existing optimal control design techniques to allow their use in designing systems of the class considered and to compare these modified design techniques to the new frequency domain design (FDD) technique.

The key step in the FDD technique is the reduction of the open loop frequency domain transfer function of the multirate system from sums of transfer functions to a single term. Once the open loop transfer function of the characteristic equation is reduced to a single term, classical frequency domain design techniques applicable for single-rate sampling may be used to design the system. A complete procedure for the FDD method is developed.

The existing design techniques which were modified are: (1) Dellon's optimal control technique, and (2) Murtuza's specific optimal control technique. The modifications to Dellon's method included: (1) inclusion of terminal cost, (2) a general computational procedure to solve for the H matrix of the controller, and (3) changes of the problem formulation which are necessary to allow application of this method to multirate systems. The modifications to Murtuza's method involved discretizing the procedure (Murtuza's procedure is given only for continuous systems) and finding the initial parameters with which to start the procedure.

The three methods are applied to the design of a guidance controller for the Apollo spacecraft which is modeled by a sixth order set of linearized difference equations.

A comparison of the three methods is made on the basis of general aspects of these methods and specific results of the Apollo design problem. Based on the Apollo problem, it is found that Dellon's method results in a controller that requires 2000 times more computer storage and four times more computer operations per sample than controllers resulting from the FDD technique or the method of Murtuza. However, Dellon's method is more straightforward in an engineering sense, and requires less than 1/20 the computer time to accomplish the design compared to Murtuza's method based on the Apollo problem. The FDD method requires the least computer design time of all the methods but requires root locus manipulations.

The controllers resulting from all three methods displayed numerical transients in the first 20 samples of operation due to controller initialization which could not be ideal because of inaccessible states of the Apollo spacecraft. The effects of bias error in one of the controller inputs is also presented.

It is shown that the system resulting from Dellon's method may be reduced to a single rate system with no degradation in the system cost. Only Murtuza's method and the FDD method seem to exploit the use of multirate sampling to increase the efficiency of the controller.

The results of the FDD method are similar to those obtained by Murtuza's method. The two methods differ in that Murtuza's method is carried out by optimization in the time domain while the FDD procedure is a frequency domain approach. The FDD procedure has the advantages of no difficulty in starting the design procedure and use of designers' intuition at all stages of the design.

CHAPTER I

INTRODUCTION

The decreasing cost of digital control equipment is permitting the increased use of digital computers for control of many analog processes. When the frequency content of several process outputs used for control is substantially different, multirate sampling can be used to reduce computer capacity in the digital control system. Multirate sampling is also often necessary because of the presence of other digital equipment, such as sensors operating at different sample rates.

The work in this dissertation is concerned with the problem of designing multirate sampled data control systems. A new design procedure is developed which allows multirate system design in the frequency domain. This was previously possible only for simple one loop multirate systems. The new frequency domain design technique is compared to two existing design techniques which are applicable to multirate systems of the class considered. Each technique is used to design a multirate controller for guidance of the Apollo spacecraft (which is modeled by a sixth order linear difference equation). The resulting controllers are discussed, and it is shown that the controller resulting from the new frequency domain design technique compares favorably in many aspects with the controllers resulting from the existing design techniques.

History of the Problem

The literature relating to the design of multirate sampled data systems can be reviewed conveniently under the three headings: Analysis of multirate sampled data control system; Design of single-rate sampled data control systems; and Design of multirate sampled data control systems.

Analysis of Multirate Sampled Data Control Systems

The analysis of multirate sampled data control systems has been investigated in great detail. Three different methods can be found in the literature, namely: state space approach, time domain switch decomposition, and frequency domain decomposition.

Kalman and Bertram [1]-[3] present a state space approach to the analysis of nonconventional sampled data systems, which includes multirate systems. This method results in obtaining the state transition matrix of the complete system, from which the discrete closed loop transfer function of the system may be obtained.

Kranc [4]-[6] develops methods of analyzing multirate systems by replacing samplers operating at various rates by equivalent sampler and delay configurations which contain sampling at only one rate. The modified z-transform is used to handle the effects of the time delays.

Coffey and Williams [7] introduce a frequency decomposition method which makes use of identities for expressing a pulsed transfer function with respect to a slow sampling rate in terms of a related pulsed transfer function with respect to a higher sampling frequency. This method requires that the system equations be written in the s-domain and then transformed into the z-domain.

Of the above three approaches, the state space method will apply to the largest class of systems. Nonsynchronized sampling and periodically varying sample period sampling may be analyzed by this method as well as multirate sampling in which the ratio of all the sample periods, taken two at a time, is a rational number.

Jury [8] shows the equivalence of the frequency decomposition and the time decomposition methods. A procedure for writing the system equations, using frequency decomposition, directly in the z-domain is outlined.

Design of Single-Rate Sampled Data Control Systems

The objective of this section is to present a brief review of the classical methods which can be used for single-rate systems, and to present several standard references for these methods. The classical methods cannot be used directly for the design of multirate sampled data control systems. One contribution of this research, however, is to provide a technique which makes it possible to use these methods and, hence, this review seems appropriate.

1. Nyquist Plot [9] -- For the design of discrete single loop feedback systems, the method of Nyquist used for continuous systems is directly applicable. $G(z)$, the plant transfer function, is plotted in the complex plane as a function of z along the Nyquist path in the z-plane. The relative and absolute stability of the uncompensated system can be observed from this plot. The effect of digital compensation can be determined from the plot simply by multiplying $G(z)$ by $H(z)$, the transfer function of the digital compensator.

2. Bode Plot/Nichols Plot [10] -- These plots are obtained through the use of the bilinear transformation $z = \frac{r + 1}{r - 1}$ or $z = \frac{1 + w}{1 - w}$ where r and w are complex variables. Either of the transformations maps the interior of the unit circle in the z plane into the left half of the r or w plane. The periodic nature of the system pole location exhibited

in the z plane with increasing frequency is not evident in the r or w plane representation. After making one of these substitutions into G(z) and H(z), the system design is carried out as for continuous systems. Design specifications are usually given by: position, velocity, or acceleration constant, phase margin, gain margin, resonant peak frequency, and system bandwidth. These specifications are readily observed from these plots and H(z) may be specified.

3. Root Locus/Root Contours [11] -- Root locus and root contours can be drawn in the z-plane just as in the s-plane for continuous systems. The plots are used in much the same way for sampled-data system design as for continuous system design. One difference is that pole and zero locations are not as easily related to time domain specifications (overshoot, rise time, damping, etc.) in the z-plane as in the case of the s-plane. Phase lead and phase lag network parameters can be obtained by use of the root locus methods. More sophisticated compensation techniques may also be designed using root locus.

4. Truxal's Synthesis Method [12],[13] -- This method assumes the positions of the desired dominant closed loop poles and zeros are known in the z-plane. The inverse root locus is used to find the open loop poles and zeros of the G(z) H(z) transfer function. H(z), the digital compensator transfer function, is then determined.

5. Minimal Control System Design [14],[15] -- There exists a frequency domain scheme to obtain the digital compensator which causes the system to respond the fastest for a given input signal. The procedure for obtaining the desired compensator involves solving for the digital compensator transfer function in terms of the plant transfer function,

G(z), and a system transfer function of a known form.

Design of Multirate Sampled Data Control Systems

Three classes of design methods for multirate systems are found in the literature: (1) design of single loop systems, (2) design of multiloop systems, (3) design of multivariable systems. Each is reviewed below, pointing out each method's advantages and disadvantages.

1a. Method of Kranc [5],[16] -- This method applies to systems of the form shown in Figure 1. A certain closed loop transfer function $K(z_n)$ is assumed to be desired and known.

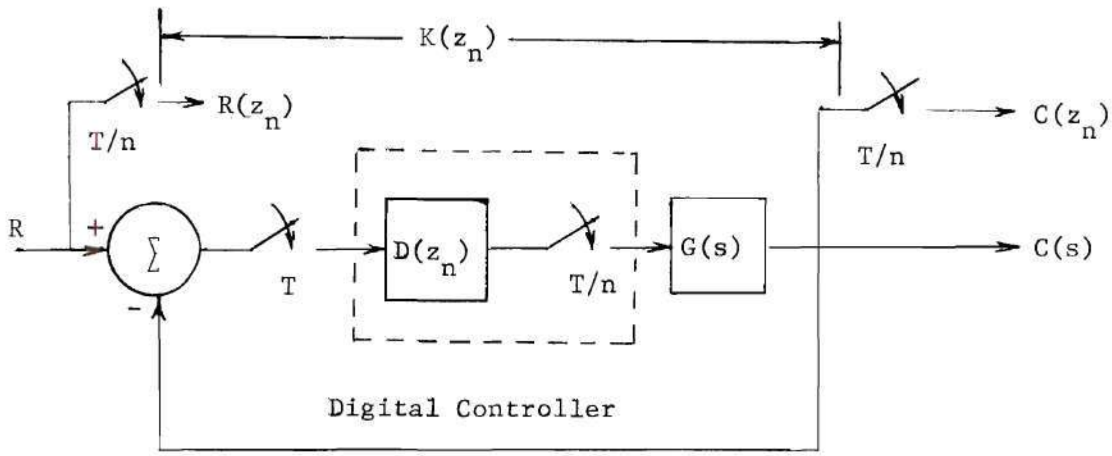


Figure 1. Structure of Systems Considered by Kranc.

The system equations are

$$C(z_n) = \frac{R(z_n^n)\ D(z_n)\ G(z_n)}{1 + Z\ \{D(z_n)\ G(z_n)\}} \tag{1.1}$$

$$D(z_n) = \frac{C(z_n)}{R(z_n^n)} = \frac{D(z_n)\ G(z_n)}{1 + Z\ \{D(z_n)\ G(z_n)\}}$$

Where $z_n = e^{sT/n}$ and $Z\{\cdot\}$ is the z transform of $D(z_n)\,G(z_n)$ with respect to $z = e^{sT}$. It is possible to solve for the desired transfer function of the digital controller in terms of $K(z_n)$ and $G(z_n)$ to obtain

$$D(z_n) = \frac{1}{G(z_n)} \; \frac{K(z_n)}{1 - K(z_n^n)} .$$

Following certain guidelines, a useful digital controller can be synthesized. Since this method has not been generalized, it is limited to single loop control systems with only one controller.

1b. Method of Knowles and Edwards [17] -- This method also considers the single loop subrate sampled data control system of Figure 1. The characteristic equation of the system (see (1.1)) is

$$1 + Z\,\{D(z_n)\,G(z_n)\} = 0$$

This equation can be expressed using frequency decomposition [8] as

$$1 + 1/n \sum_{k=0}^{n-1} D(z_n e^{j2\pi k/n})\; G(z_n e^{j2\pi k/n}) = 0 \qquad (1.2)$$

If the open loop system is assumed to be low pass with respect to the lower sampling frequency (1/T), then the sum above can be approximated by the k = 0 term. The classical design techniques can then be applied directly to the design problem.

This method has not been generalized, so it is limited to single loop control systems with only one controller, and two sample rates.

The disadvantage of this design method is that the sample rates must be high with respect to the natural frequencies of the system. This constraint may result in unnecessarily high sampling rates with resulting controller inefficiency.

1c. Method of Crisp and Phillips [18],[19] -- The method introduced by Phillips is based on observations of the characteristic equation of a single loop subrate system (1.2). It is shown that the sum

$$\sum_{k=0}^{n-1} D(z_n e^{j2\pi k/n})\, G(z_n e^{j2\pi k/n}) \tag{1.3}$$

is periodic in ω with period $2\pi/T$ radians per second, where $z_n = e^{j\omega T/n}$. Therefore only ω in the range $\{-\pi/T,\pi/T\}$ need be considered. Further it is noted that each term of the sum is only the basic term, $G(z_n)\, D(z_n)$, shifted in frequency. That is, for $\omega \varepsilon \{-\pi/T,\pi/T\}$, the first term of the sum is $D(e^{j\omega_1 T/n})\, G(e^{j\omega_1 T/n})$ where $\omega_1 = \omega$. The second term is $D(e^{j\omega_2 T/n})\, G(e^{j\omega_2 T/n})$ where $\omega_2 = \omega + 2\pi/T$. If $G(z_n)$ contains resonant modes for frequencies $\omega > \pi/T$, the manner in which these resonant modes affect the multirate system's frequency response can be seen. If compensation is necessary, $D(z_n)$ may be constructed to affect the desired change. Crisp makes similar observations in the w_n-plane after transforming the open loop transfer function (1.3) using the bilinear transformation, $z_n = \frac{1 + w_n}{1 - w_n}$. The disadvantage of the above design method is that it is qualitative rather than quantitative. The classical design techniques cannot be applied directly to a characteristic equation in which the open loop transfer function appears as a sum of terms.

Crisp and Phillips have applied the design method only to single loop systems in which the ratio of the sample periods is an integer (or its reciprocal).

2. Method of Coffey [7],[22] -- A method which can be applied to multiloop, multirate sampled data control systems of the structure shown in Figure 2 has been introduced by Coffey. The system is further restricted as follows: (1) The sampler with the longest period, T^*, must immediately precede the plant, and (2) the ratio $\frac{T^*}{T_i}$ must be an integer, where T_i is any sample period in the system.

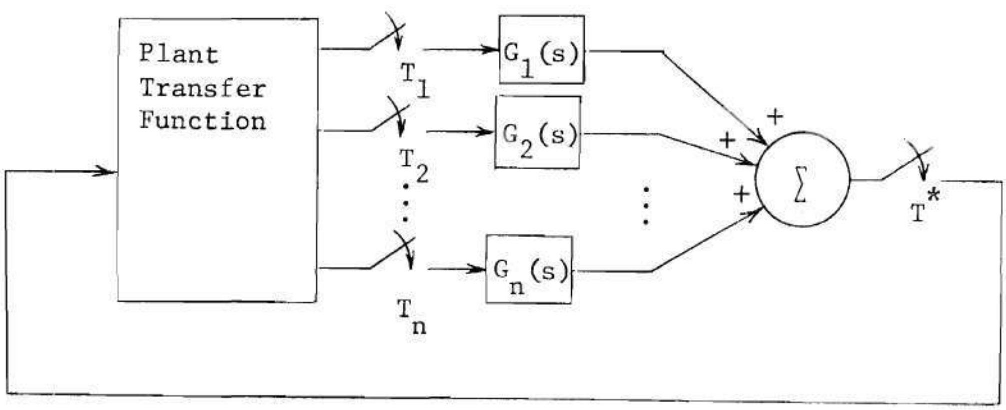


Figure 2. Structure of Systems Considered by Coffey.

The design method utilizes frequency decomposition in writing the system equations. No attempt is made to simplify these equations beyond the point of having all the terms of the equation expressed in the same variable, z. The resulting equations, for all but trivial systems, are very complex. Given a fixed controller structure, the procedure is structured so a large digital computer can be programmed to find the controller parameters which result in the best least squares fit of a desired open loop phase/gain plot. A gradient search algorithm is used to optimize the controller parameters.

Three disadvantages of this design approach are: (1) the class of systems considered is very restricted; (2) the structure of the controller must be specified before the system parameters are optimized. The relation between the resulting system's frequency domain specifications and different controller structures can only be discovered by repeated design attempts using different structures; (3) the design engineer gains little insight into how the controller structure and parameters affect the frequency domain specifications of the resulting system. This method has the advantage of being capable of application to very complex systems.

3a. Optimal Control Approach -- Multivariable, multirate sampled data control systems can be designed using existing optimal control theory. Existing design approaches are formulated using the state variable representation of the system. It has been shown [1]-[3] that the state transition matrix of a multirate system is complicated by being periodically time varying. The class of systems considered is further complicated by having a plant with inaccessible states in general.

Ferguson and Rekasius [23] developed the theory to design optimal linear time varying control systems with incomplete state measurements for continuous systems. Dellon [24],[25] extended this theory to include discrete control systems with unstable plants using a quadratic cost functional over a finite time interval. The plant is assumed to be uniformly completely observable. Dellon's resulting controller includes an estimation procedure which involves solving (n-m) linear differential equations, where n is the order of the plant and m is the number of independent plant outputs. The estimated state vector is then multiplied by time varying gains, which must be stored in the controller, to obtain the control signal for the plant.

3b. Specific Optimal Control -- Optimal design techniques which assume the controller has a specific structure with one or more parameteres free for adjustment are found in the literature for multivariable control systems. This design approach is similar to that of Coffey [2] with the exception that the design criteria is specified in the time domain instead of the frequency domain. Several design techniques have been developed [26]-[29] which assume that the controller structure consists of fixed or time varying gains. Murtuza [30] develops the theory to include a linear dynamic controller structure. All of the above work is for continuous systems.

Statement of the Problem

The review of the literature, summarized above, indicates that existing design procedures for multirate, multiloop sample data systems with any degree of generality must be carried out using optimal control techniques.

The applicable methods can be classified as general optimal techniques and specific optimal techniques. The literature survey indicates that the general optimal control technique of Dellon [24] and the specific optimal technique of Murtuza [30], (appropriately modified to apply to sample data systems), are representative of the best that is available.

Although frequency domain equations may be written which describe the dynamics of multirate sampled data systems [7], the frequency domain characteristic equation of a multirate loop involves a sum of transfer functions. Since classical frequency domain design techniques (e.g., Root Locus, Nyquist Plots, etc.) depend on the open loop transfer function in the characteristic equation being expressed as a single term, it is not possible to apply these classical techniques directly to multirate systems.

The primary objectives of this research are: (1) to develop a general procedure for frequency domain design of a large class of multirate, multiloop sample data systems based on a technique developed to reduce the open loop transfer function to a single term, and (2) to compare the new procedure to the methods of Dellon and Murtuza as representative of the best existing procedures applicable to the same class of systems.

CHAPTER II

DEVELOPMENT OF THE NEW FREQUENCY DOMAIN DESIGN (FDD) PROCEDURE

As mentioned in Chapter I, the key step in carrying out the new frequency domain design procedure involves reducing the form of the open loop transfer function of the multirate system, one loop at a time to a single transfer function. In this chapter, the class of systems to be considered is defined, and then the method of reducing the multirate transfer function is developed.

Definitions to be Used in this Dissertation

The following definitions will be used throughout this dissertation:

1. Difference equations will be used to describe the dynamics of both analog and discrete equipment. In describing analog equipment, the solution $x(k)$ of a difference equation such as $x(k+1) = A\,x(k)$ for $k = 0, 1, 2, \ldots$, will represent the analog variable $x(t)$ in the time interval $(kT, kT + T]$. In such cases, T is referred to as the "period of the difference equation" and it is determined by the samplers associated with the analog equipment.

2. The term T_k^* will be the period of a sampler with the longest period in the k^{th} loop of the system. The period of the i^{th} sampler in the k^{th} loop will be T_k^i. The samplers are numbered in the clockwise direction starting with one as the first sampler with period T_k^* to the right of the summing junction in the forward path of the k^{th} loop.

Class of Systems to be Studied

The systems to be studied in this research are linear time invariant multiloop multirate sampled data systems whose structure is shown in Figure 3.

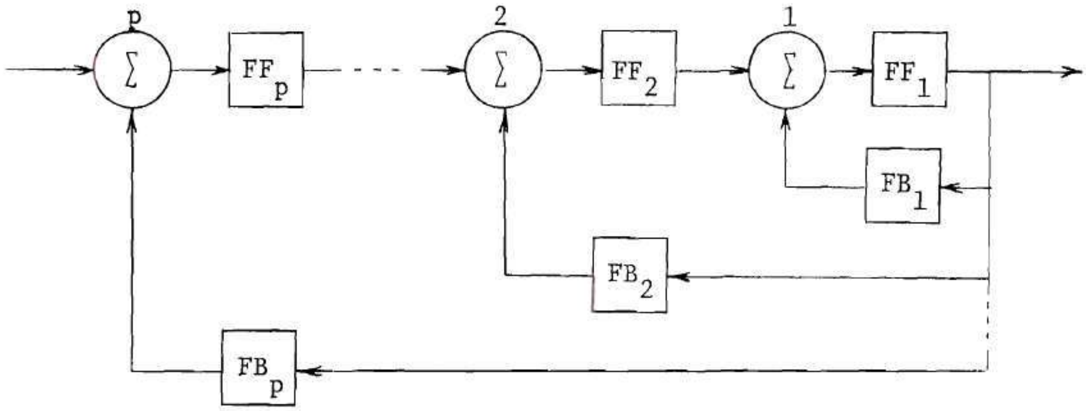


Figure 3. Structure of Systems to be Studied.

Each of the boxes shown in Figure 3 contains a series of single input, single output components which may be analog or discrete equipment. These components are separated by ideal samplers[1] which may have different periods. It is assumed that the dynamics of both analog and discrete components can be described by a difference equation with period T of the form

$$x(k + 1) = A\,x(k) + bu(k) \tag{2.1}$$

$$y(k) = c\,x(k) + du(k) \qquad \text{for } k = 0,1,2, \text{---} \tag{2.2}$$

where

x is an n dimensional state vector

u is the scalar input

y is the scalar output

A,b,c, and d are constant matrices

---

[1] An ideal sampler sampling x(t) at kT produces x(kT).

T is the sample period of either the input or output sampler of this component - whichever is smaller.

For example, box FF_1 may contain two components separated by samplers as shown in Figure 4. The difference equations describing

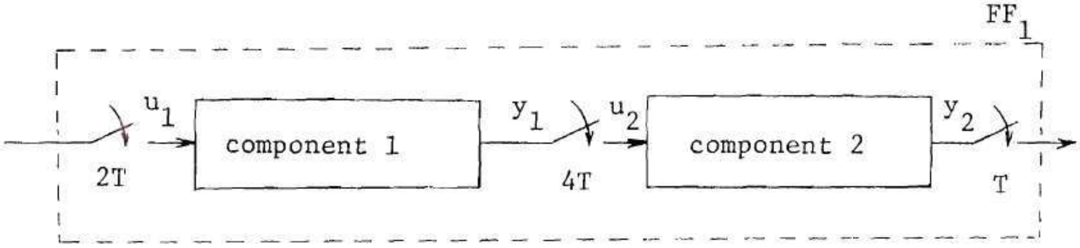


Figure 4. Example of System Structure.

the dynamics of the two components are of the form

$$\left.\begin{aligned} x_1(k+1) &= A_1x_1(k) + b_1u_1(k) \\ y_1(k) &= c_1x_1(k) + d_1u_1(k) \end{aligned}\right\} \begin{array}{l}\text{difference equation of component 1} \\ \text{with period 2T}\end{array}$$

$$\left.\begin{aligned} x_2(k+1) &= A_2x_2(k) + b_2u_2(k) \\ y_2(k) &= c_2x_2(k) + d_2u_2(k) \end{aligned}\right\} \begin{array}{l}\text{difference equation of component 2} \\ \text{with period T.}\end{array}$$

In this case only every fourth input value to component 2 will be non-zero. A zero order hold may be included in the difference equations of each component if desired [20].

The following restrictions are placed on the system:

1. The ratio $\frac{T_k^*}{T_k^i}$ must be an integer. This requires the longest sample period of the k^{th} loop to be a multiple of each sample period of this loop. However, notice that the ratio $\frac{T_k^i}{T_k^j}$ may be a rational number.

2. The ratio $\frac{T^*_{k+1}}{T^*_k}$ must be an integer. This requires the longest sample period in the $(k+1)^{th}$ loop to be a multiple of the longest sample period of the k^{th} loop. Notice that T^*_{k+1} may equal T^*_k.
3. At least one sampler with the longest period in the k^{th} loop must appear in box FF_k.

Simplification of the Open Loop Transfer Function

The crucial step in the new design procedure is the reduction of the open loop frequency domain transfer function of the multirate system to a single transfer function. This is done one loop at a time as shown in Figure 5.

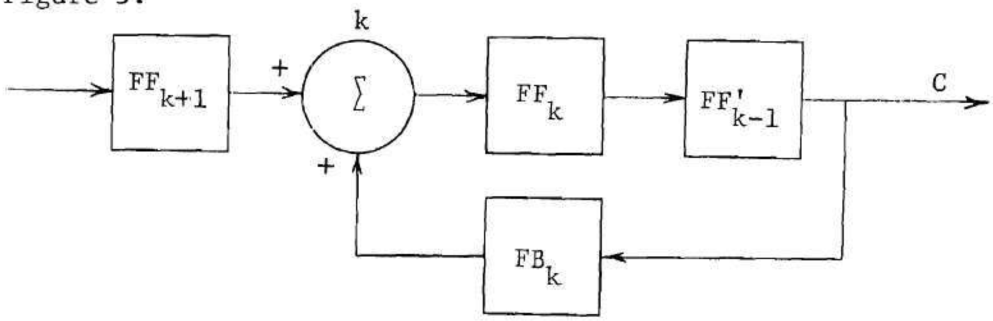


Figure 5. Structure of the k^{th} Simplified Loop.

The transfer function(s) resulting from the reduction of the first (k-1) loops of the system to open loop form are contained in box FF'_{k-1}. Let Y(z) be the frequency domain output of the sampler in FF_k which is nearest to the right of the k^{th} summing junction in the forward path and which has the longest period, T^*_k, in the k^{th} loop. Let R(z) be the frequency domain output of the sampler in FF_{k+1} which is nearest to the left of the k^{th} summing junction and which has period $T \geq T^*_k$. The selection of Y and R is shown in detail later in this section. Further it is shown that

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + H(z)} \tag{2.3}$$

where $z = e^{sT_k^*}$, and G(z) and H(z) are obtained by the procedure described later in this section.

Once (2.3) has been verified, it is a straightforward procedure to reduce the k^{th} loop shown in Figure 5 to the open loop form shown in Figure 6.

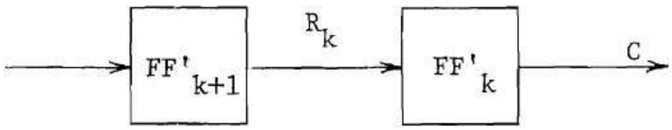


Figure 6. Simplified Open Loop Configuration.

The box FF'_{k+1} in Figure 6 contains all the components in the box FF_{k+1} shown in Figure 5 that are to the left of R(z).

The procdure outlined above is repeated one loop at a time until the entire system is reduced to the form of one open loop transfer function.

The reduction of a typical loop is carried out below on the system shown in Figure 7.

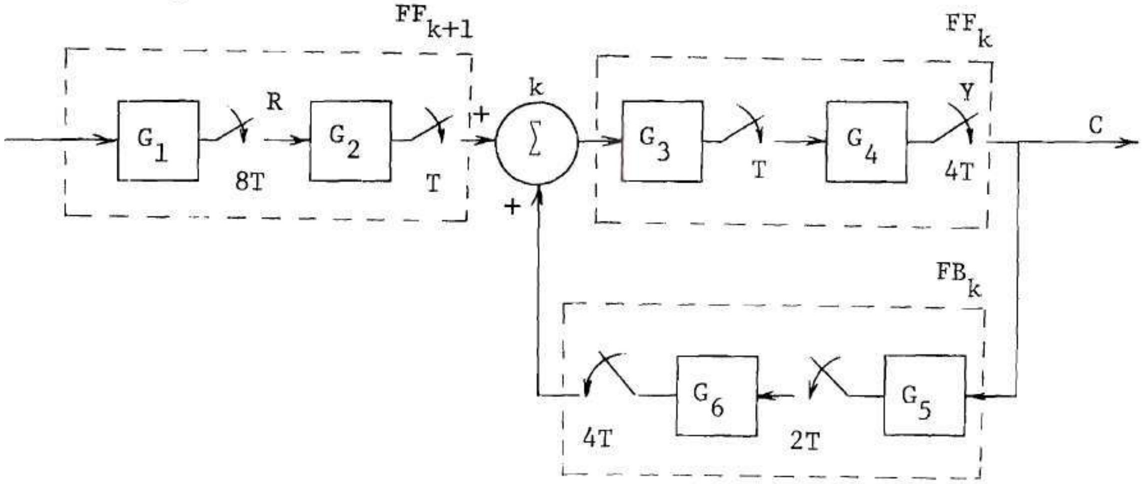


Figure 7. Structure of k^{th} Simplified Loop.

Y and R are specified as previously mentioned. Standard techniques [8] yield

$$Y(z_4) = R(z_4)\ \{\frac{1}{4}\sum_{a=0}^{3} G_4(z_1 e^{j2\pi\frac{a}{4}})\ G_3(z_1 e^{j2\pi\frac{a}{4}})\ G_2(z_1 e^{j2\pi\frac{a}{4}})\} +$$

$$Y(z_4)\ \{\frac{1}{4}\sum_{b=0}^{3} G_4(z_1 e^{j2\pi\frac{b}{4}})\ G_3(z_1 e^{j2\pi\frac{b}{4}})\} \left\{\frac{1}{4}\sum_{c=0}^{3} G_6(z_1 e^{j2\pi\frac{c}{4}})\right.$$

$$\left.\left[\frac{1}{2}\sum_{d=0}^{1} G_5(z_1 e^{j2\pi(\frac{d}{2}+\frac{c}{4})})\right]\right\} \tag{2.4}$$

where $z_1 = e^{j2\pi T}$ and $z_4 = e^{j2\pi(4T)}$, which describes the loop in the z domain. The characteristic equation for the k^{th} loop, as obtained from (2.4), is given as

$$1 - \underbrace{\{\frac{1}{4}\sum_{b=0}^{3} G_4(z_1 e^{j2\pi\frac{b}{4}}) G_3(z_1 e^{j2\pi\frac{b}{4}})\}}_{\text{term A}}\ \{\frac{1}{4}\sum_{c=0}^{3} G_6(z_1 e^{j2\pi\frac{c}{4}}) \underbrace{[\frac{1}{2}\sum_{d=0}^{1} G_5(z_1 e^{j2\pi(\frac{d}{2}+\frac{c}{4})})]}_{\text{term B}}\} = 0 \tag{2.5}$$

To use classical design techniques (e.g. Root Locus, Nyquist Plot, etc.), the open loop transfer function of (2.5) must appear as a single transfer function, not as sums as shown in (2.5). The reduction of the sums of transfer functions in (2.5) to a single transfer function is accomplished in the following way.

Consider term A in (2.5). It is seen in Figure 7 that G_4 represents a system component (located in box FF_k) whose dynamics may be represented by a difference equation with period T. Standard techniques are available to transform the representation from difference equations to a z domain transfer function [32]. Since the difference equation has period T, the z domain transfer function will be in terms of z_1. Thus, the first term of the sum over b in (2.5) will be $G_4(z_1)\ G_3(z_1)$, a product of z domain transfer functions in the z_1 variable. Since the difference equations describing the dynamics of components G_3 and G_4 are linear constant coefficient (time invariant), the z domain transfer function is of the form

$$G_4(z_1)G_3(z_1) = \frac{K_1(z_1+\beta_1)\ (z_1+\beta_2)\cdots(z_1+\beta_\zeta)}{(z_1+\gamma_1)\ (z_1+\gamma_2)\cdots(z_1+\gamma_\rho)}. \tag{2.6}$$

It is seen that the common denominator of the sum in term A of (2.5) is

$$(-z_1^4+\gamma_1^4)(-z_1^4+\gamma_2^4)\cdots(-z_1^4+\gamma_\rho^4). \tag{2.7}$$

If term A is multiplied by this common denominator, it may be represented as

$$K_1 \sum_{b=0}^{3} (z_1+a_1 e^{j2\pi\frac{b}{4}})\ (z_1+a_2 e^{j2\pi\frac{b}{4}})\cdots(z_1+a_n e^{j2\pi\frac{b}{4}})\ e^{j2\pi\frac{bm}{4}} \tag{2.8}$$

where $m = \rho - \zeta$, $n = 4\rho - m$, and a_i are complex numbers related to the poles and zeros of $G_4(z_1)\ G_3(z_1)$. Expanding (2.8), it is seen to equal (see Appendix)

$$4K_1(\alpha_{n-4(\rho-\eta)} z_1^{4(\rho-\eta)} + \alpha_{n-4(\rho-\eta-1)} z_1^{4(\rho-\eta-1)} + \cdots + \alpha_{n-4} z_1^4 + \alpha_n) \tag{2.9}$$

where α_j are the sums of the products of all combinations of a_i terms taken j at a time. The value η is the smallest integer such that $\eta \geq \frac{m}{4}$. Since $z_1^4 = z_4$, term A of (2.5) may be rewritten as

$$K_1 \frac{(\alpha_{n-4(\rho-\eta)} z_4^{\rho-\eta} + \alpha_{n-4(\rho-\eta-1)} z_4^{\rho-\eta-1} + \ldots + \alpha_{n-4} z_4 + \alpha_n)}{(z_4 + \gamma_1^4)\ (z_4 + \gamma_2^4)\ \ldots (z_4 + \gamma_\rho^4)}. \tag{2.10}$$

It is seen that the poles of (2.10) are the same as the poles of (2.6), while the number of zeros is different and their values are different in general.

Now consider term B in (2.5). It may be rewritten as

$$\frac{1}{2} \sum_{d=0}^{1} G_5(z_1 e^{j2\pi(\frac{d}{2} + \frac{c}{4})}) = \frac{1}{2} \sum_{d=0}^{1} G_5(z_1^* e^{j2\pi\frac{d}{2}}) \tag{2.11}$$

where $z_1^* = z_1 e^{j2\pi\frac{c}{4}}$ for a given integer c. Using the same technique employed in term A, (2.11) is seen to be equal to a single transfer function in the variable z_1^*

$$\frac{1}{2} \sum_{d=0}^{1} G_5(z_1 e^{j2\pi(\frac{d}{2} + \frac{c}{4})}) = G_5'(z_1^*) = G_5'(z_1 e^{j2\pi\frac{c}{4}}). \tag{2.12}$$

The sum over c in (2.5) becomes

$$\frac{1}{4} \sum_{c=0}^{3} G_6(z_1 e^{j2\pi\frac{c}{4}})\ G_5'(z_1 e^{j2\pi\frac{c}{4}}) \triangleq \overline{G_6 G_5}(z_4) \tag{2.13}$$

which may be reduced to a single transfer function in the variable z_4 by using the technique employed in term A again. Thus (2.4) becomes

$$Y(z_4) = R(z_4)\ \overline{G_4 G_3 G_2}(z_4) + Y(z_4)\ \overline{G_4 G_3}(z_4)\ \overline{G_6 G_5}(z_4) \tag{2.14}$$

and

$$\frac{Y(z_4)}{R(z_4)} = \frac{\overline{G_4G_3G_2}(z_4)}{1 - \overline{G_4G_3}(z_4)\ \overline{G_6G_5}(z_4)} . \tag{2.15}$$

Now the k^{th} loop may be represented as shown in Figure 8.

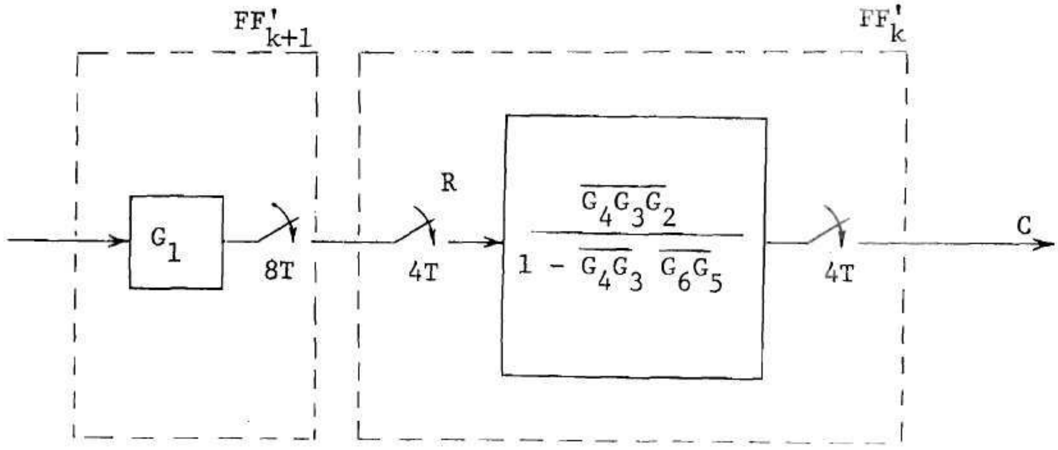


Figure 8. Simplified Open Loop Configuration.

This is seen to be the form shown in Figure 6. The procedure developed above is applied one loop at a time until the reduction of the complete system is accomplished.

CHAPTER III

DETAILED DESCRIPTION OF USE OF FDD PROCEDURE

The basic procedure for using the design technique developed in Chapter II is described in this chapter. The portion of this procedure which as been implemented as a computer program is described. The basic aspects of the new design procedure are illustrated in the final section of this chapter, using a low order system as an example.

Using the Technique

The FDD design procedure makes possible the design of a multirate system using the classical single rate design methods. In general, the structure of the system dictates the classical design technique which is best to use. For example, Nyquist Plot, Nichols Chart, Bode Plot, or Root Locus may be chosen for the design of a single loop system. Root Contours offer design flexibility when working with multiloop systems. Using Root Controus, the system may be designed one loop at a time, since characteristic equations may be written, as developed in Chapter II, one loop at a time.

The basic procedure for carrying out the FDD design procedure is outlined below:

1. Choose the classical design procedure that is best suited for the system structure and the form of the design specifications. Root Contours are suggested for multiloop systems, since the new design procedure

is to be performed one loop at a time. However, if design specifications other than pole location are available on a loop by loop basis (e.g. phase/gain plots), other classical methods may be used for multiloop systems.

2. Obtain the system difference equations or the z domain transfer functions of the components of the system. For analog components whose system equations are continuous rather than discrete, their equations may be put in state equation form and the matrix exponential evaluated to obtain the state transition matrix. The difference equations describing the system may then be written. Various computerized routines exist which will accomplish this complete task [32].

3. Use the procedure, described in Chapter II, for simplifying the open loop transfer function to reduce the open loop transfer function of the first loop to a single term. This procedure may be implemented as a computer program as described later in this chapter. Although the poles of the open loop transfer function remain the same after the reduction to a single term, the new zeros are a function of both the original poles and zeros of the open loop transfer function of the first loop.

4. Using the reduced open loop transfer function, apply the selected classical design technique to design the controller for the first loop, as though the system were a single rate system.

5. Having designed the first loop, find the characteristic equation for the first two loops. Simplify the open loop transfer function in this equation again using the procedure developed in Chapter II. Using the reduced open loop transfer function, design the second loop controller using the chosen classical design technique.

6. Iterate the above step until all controller parameters in all loops have been specified.

7. If the desired system specifications have not been met, return to step 3 and iterate the complete design again, just as would be done if the system were single rate. Although the design procedure outlined above may be tedious for higher order systems, this FDD procedure adds a new dimension to previously existing multirate, multiloop design techniques, namely design in the frequency domain.

The procedure for reducing the open loop transfer function of a multirate system to one term, as mentioned above in step 3, may be computerized. The flow diagram of a computer program written for this purpose is shown in Figure 9. Since the design procedure is a one-loop-at-a-time procedure, the program is designed to calculate the zeros of the open loop transfer function of only one loop.

To implement the computer program, the k^{th} loop is divided into segments, where a segment is the transfer function of the portions of the system between samplers with period T_k^*, the longest sample period in the k^{th} loop. The simplified open loop transfer function is then the product of the loop segments. Each segment is simplified separately by the computer program, and the zeros of each segment are a function of only the original poles and zeros of that segment.

It is seen that if the controller is originally situated in a segment by itself, its difference equation may be originally written with period T_k^*, and there is no reason to use the computer program to calculate the open loop zeros of this segment. In application and design using this FDD design technique, an isolated controller segment is particularly easy

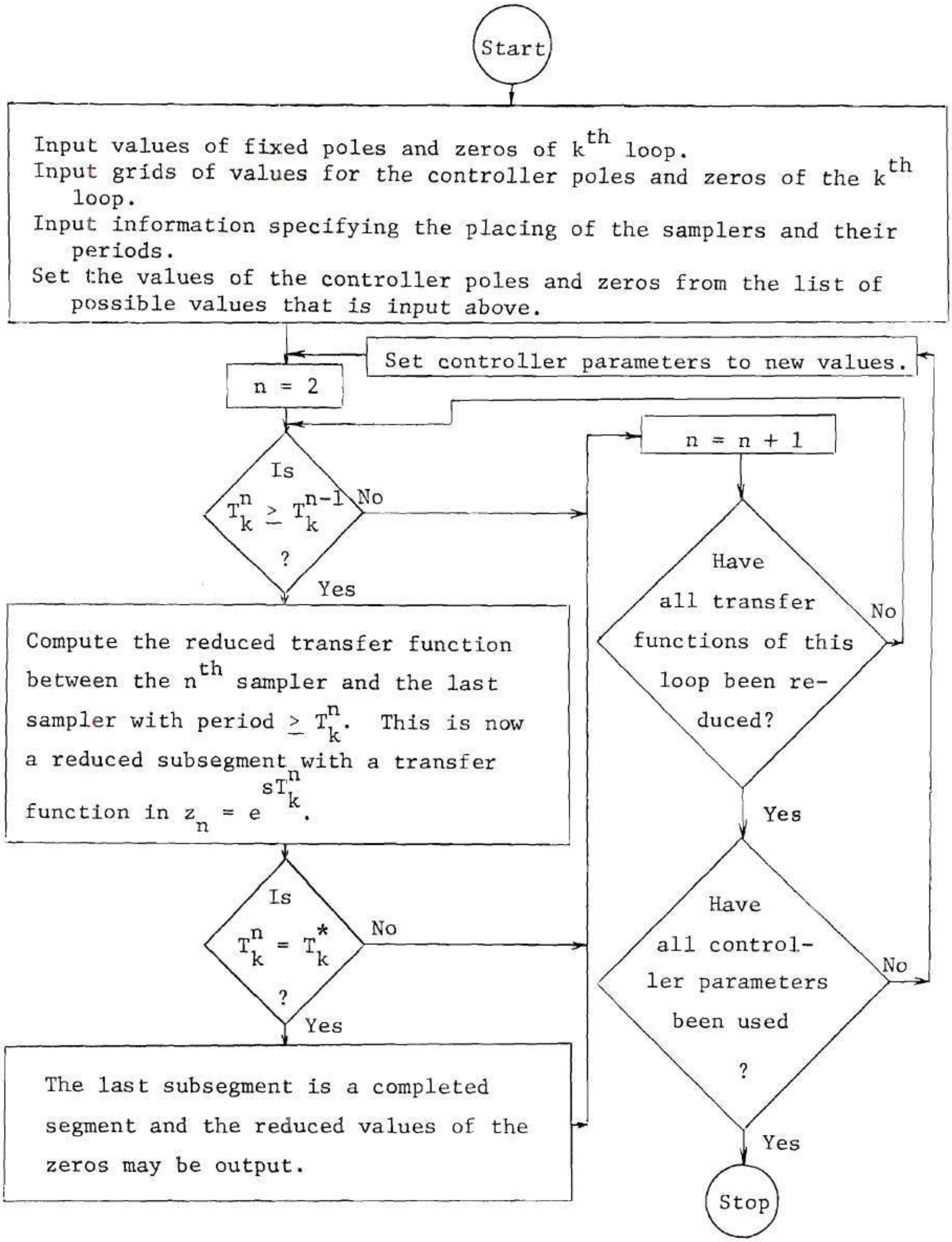


Figure 9. Flow Graph of Computer Program Used to Simplify the Open Loop Transfer Function

to deal with since the zeros of the controller can be specified directly without use of the computer program to find them.

If the controller occurs in a segment with another transfer function, the desired segment zeros (including the reduced controller zeros) can be obtained as follows. Input the fixed poles and zero values into the computer program for the controller segment. Then input a grid of values which will be considered for the controller poles. Then input a grid of controller zero values which cover the unit circle in the z plane. The computer program will now compute the resulting simplified controller segment zeros (thus, including the controller zeros) for each parameter value of each grid point. Preliminary design of this loop should indicate the approximate controller pole and zero values desired of the reduced transfer function. To refine the design parameters, a new grid of controller values which form a finer grid in certain areas of interest of the z plane may now be input to the computer program. This procedure may be iterated to obtain more accurate parameter values.

Although the controller segment will always have a certain number of zeros that are free to be specified (i.e. controller zeros), it is usually the case that the simplified controller zeros cannot be placed arbitrarily. For example, if the controller has a pole at zero, the simplified controller segment transfer function must have a zero at the origin of the z plane. In any case, the grid search method described above will give the designer a good indication of the possible positions and arrangements of controller segment zeros.

Example of the Use of the Procedure on a Low Order System

Consider the single loop multirate system shown in Figure 10.

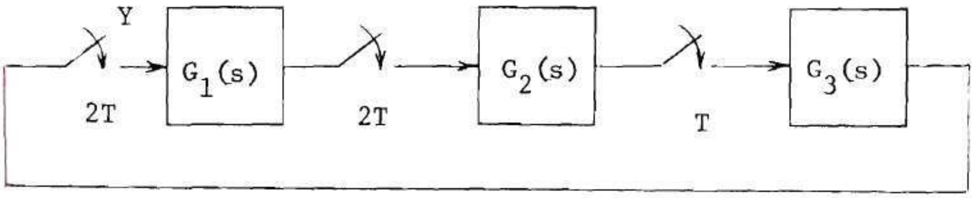


Figure 10. Example of a Single Loop System.

Given the design specifications for this system and the equations describing the dynamics of this system, it is straight forward to carry out the first two steps of the design procedure. This includes choosing a desirable classical design procedure and obtaining the difference equations which describe the dynamics of the three components of the system shown in Figure 10.

Step 3 of the procedure may be carried out by using the previously mentioned computer program. This step is carried out in detail below to illustrate the basic procedure used by the computer program.

An expression describing the dynamics of the system shown in Figure 10 can be written in terms of the pulsed transfer as follows

$$Y^*(s) = \frac{1}{2T}\sum_a Y(s+j2\pi\frac{a}{2T}) = \frac{1}{2T}\sum_b G_3(s+j2\pi\frac{b}{2T})\ \frac{1}{T}\sum_c G_2\left[s+j2\pi(\frac{c}{T}+\frac{b}{2T})\right]$$

$$\frac{1}{2T}\sum_d G_1\left[s+j2\pi(\frac{d}{2T}+\frac{c}{T}+\frac{b}{2T})\right]\frac{1}{2T}\sum_e Y\left[s+j2\pi(\frac{e}{2T}+\frac{d}{2T}+\frac{c}{T}+\frac{b}{2T})\right]$$

where the sums above are over $[-\infty,\infty]$ unless otherwise noted. Rewriting this equation results in the following expression

$$\frac{1}{2T}\sum_a Y(s+j2\pi\frac{a}{2T}) = \frac{1}{2T}\sum_e Y(s+j2\pi\frac{e}{2T})\ \frac{1}{2T}\sum_d G_1(s+j2\pi\frac{d}{2T})$$

$$\frac{1}{2}\sum_{f=0}^{1}\ \frac{1}{T}\sum_b G_3\left[s+j2\pi(\frac{b}{T}+\frac{f}{2T})\right]\ \frac{1}{T}\sum_c G_2\left[s+j2\pi(\frac{c}{T}+\frac{b}{T}+\frac{f}{2T})\right]. \tag{3.1}$$

This simplification is a result of the fact that a sum

$$\sum_k H\left[s+j2\pi(\frac{k}{T}+\frac{\ell}{T})\right]$$

has the same value for all finite integers l, since the sum is periodic with period T. Also the sum shown on the left

$$\sum_{k=-\infty}^{\infty} H(s+j2\pi\frac{k}{2T}) = \sum_{\ell=0}^{1}\ \sum_{k=-\infty}^{\infty} H\left[s+j2\pi(\frac{k}{T}+\frac{\ell}{2T})\right]$$

may be rewritten as shown above by breaking the left sum into two interlocking sums.

Now rewriting (3.1) using the z transform formulation; the resulting equation is shown below

$$Y(z_2) = Y(z_2)\ G_1(z_2)\ \frac{1}{2}\sum_{f=0}^{1} G_3(z_1 e^{j2\pi\frac{f}{2}})\ G_2(z_1 e^{j2\pi\frac{f}{2}}) \tag{3.2}$$

where

$$z_1 = e^{sT} \quad\text{and}\quad z_2 = e^{s2T}.$$

The above sum on f is now simplified into one term. This is illustrated using the following transfer functions.

$$G_1(z_2) = \frac{z_2 + 5}{z_2 - \frac{1}{2}}; \quad G_2(z_1) = K\frac{z_1 + a}{z_1 + b}; \quad G_3(z_1) = \frac{z_1 + 1}{z_1 + 2} \tag{3.3}$$

$$\frac{1}{2}\sum_{f=0}^{1} G_3(z_1 e^{j2\pi\frac{f}{2}})G_2(z_1 e^{j2\pi\frac{f}{2}}) = \frac{K}{2}\left|\left\{\frac{z_1 + a}{z_1 + b}\right\}\left\{\frac{z_1 + 1}{z_1 + 2}\right\} + \left\{\frac{-z_1 + a}{-z_1 + b}\right\}\left\{\frac{-z_1 + 1}{-z_1 + 2}\right\}\right|$$

$$= \frac{2K[z_1^4 + (a\text{-}ab\text{-}2a\text{-}b\text{-}2\text{+}2b)z_1^2 + 2ab]}{2(z_1^2 - b^2)(z_1^2 - 2^2)}$$

$$= K\frac{(z_2^2 + z_2(a\text{-}ab\text{-}2a\text{-}b\text{-}2\text{+}2b) + 2ab)}{(z_2 - b^2)(z_2 - 2^2)} \tag{3.4}$$

It is seen that the poles of the transfer functions given in (3.3) and the reduced multirate transfer function given in (3.4) are the same. The reduced transfer function's zeros are, however, shifted. If transfer function G_2 represents a controller with zero value of -a and pole value of -b, it is seen that the reduced transfer function's numerator must be factored for each combination of controller parameters that will be considered in the design of this loop. This system has two segments: G_1 and $\overline{G_3G_2}$. It is seen that the zeros of each segment are a function of only the poles and zeros of each segment.

The characteristic equation for this system which is obtained from (3.2) is

$$1 - K\left\{\frac{z_2 + 5}{z_2 - \frac{1}{2}}\right\}\left\{\frac{z_2^2 + z_2(-a-ab+b-2) + 2ab}{(z_2 - b^2)(z_2 - 4)}\right\} = 0 \tag{3.5}$$

Step 3 is completed by finding the open loop zeros of (3.5) for the various grid values assigned to the parameters a and b.

Step 4 involves using the simplified open loop transfer function in (3.5) in conjunction with a classical design technique to complete the design of the first and only loop of the system. This can be accomplished in a straight forward manner, as though the system were a single rate system.

Step 5 through 7 may be omitted since the system has only one loop.

CHAPTER IV

OPTIMAL TECHNIQUES APPLIED TO MULTIRATE SAMPLED DATA SYSTEMS

Two techniques exist for designing multirate sampled data control systems of the class considered: optimal control technique and specific optimal control technique. The optimal control design technique of Dellon [24],[25], and the specific optimal control technique of Murtuza [30] are described in this chapter. Various modifications that are necessary to make these methods generally applicable to multirate sampled data systems such as the Apollo guidance control system to be considered in Chapter V are also discussed.

Optimal Control Technique

The optimal control technique of Dellon and the required modifications to this technique are described in the following two sections.

Dellon's Method

This method can be used on a uniformly completely observable plant whose dynamics can be described by the n^{th} order difference equation

$$\begin{aligned} x(k+1) &= A(k) \; x(k) + B(k) \; u(k) \\ y(k) &= C(k) \; x(k) \quad \text{for } k \varepsilon [k_0, N-1], \end{aligned} \tag{4.1}$$

where

A(k) is an n x n matrix

B(k) is an n x r matrix

$C(k)$ is an m x n matrix ($m \le n$) of rank m

$x(k)$ is an n dimensional state vector

$u(k)$ is an r dimensional input vector

$y(k)$ is an m dimensional output vector

and $A(k)$, $B(k)$, and $C(k)$ are bounded. The control sequence $u^*(k)$ which minimizes the performance functional

$$J(u(k),\ x(k_0),\ k_0) = \frac{1}{2} \sum_{i=k_0}^{N-1} [y^T(i)Q(i)y(i) + u^T(i)R(i)u(i)]$$

is determined, where

$Q(i)$ is a positive definite m x m matrix

$R(i)$ is a positive definite r x r matrix.

The feedback system resulting from this design method is guaranteed to be stable. The structure of the system is shown in Figure 11.

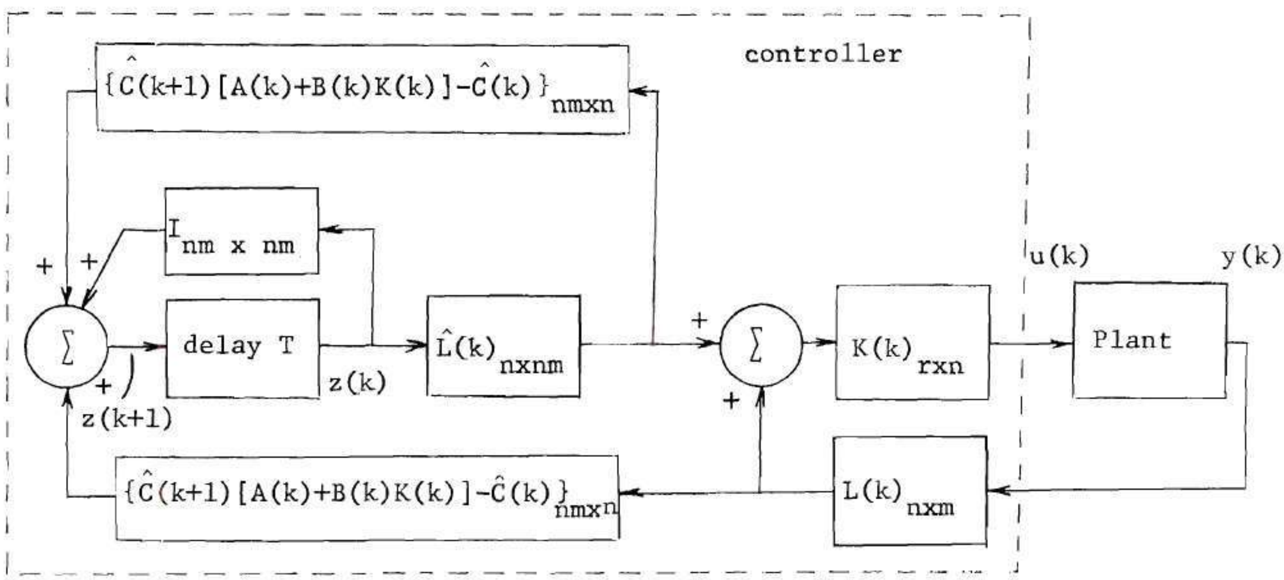


Figure 11. Structure of Dellon's Controller.

The matrices used in the controller are defined below. The $\hat{C}$ matrix is defined in terms of an H(k) matrix as

$$\hat{C}(k) = [H(k)_{nm \times m} \mid I_{nm \times nm}] \quad \text{where} \quad nm = n-m.$$

H(k) is defined by the equation

$$G(k) = A_{22}(k)_{nm \times nm} + H(k)_{nm \times m} A_{12}(k)_{m \times nm} \tag{4.2}$$

where

$$A(k) = \left[\begin{array}{c|c} A_{11}(k)_{m \times m} & A_{12}(k)_{m \times nm} \\ \hline A_{21}(k)_{nm \times m} & A_{22}(k)_{nm \times nm} \end{array}\right]$$

and G(k) is any matrix with all zero eigenvalues.

Dellon shows that the vector output equation can always be put into the form

$$y(k) = [C_1(k)_{m \times m} \mid 0_{m \times nm}]x(k)$$

where $C_1(k)$ is nonsingular for $k \in [k_o, N-1]$. The proof of this relation is carried out using a similarity transformation of (4.1).

The L and $\hat{L}$ matrices are defined as

$$L(k) = \begin{bmatrix} C_1^{-1}(k)_{m \times m} \\ \\ -H(k)_{nm \times m} \cdot C_1^{-1}(k)_{m \times m} \end{bmatrix} \qquad \hat{L}(k) = \begin{bmatrix} 0_{m \times nm} \\ \\ I_{nm \times nm} \end{bmatrix}$$

The K(k) are the optimal gains of the system which are generated from the solution of the Riccati equation

$$P(k) = A(k)^T P(k+1)[I - B(k)\{R(k) + B(k)^T P(k+1) B(k)\}^{-1} B(k)^T P(k+1)] \\ A(k) + C(k)^T Q(k) C(k) \tag{4.3}$$

where $P(N) = 0$

and $K(k) = -(R(k) + B(k)^T P(k+1)B(k))^{-1} B(k)^T P(k+1)A(k).$ (4.4)

The cost resulting from using the above controller is in general larger than the cost for a system in which all plant states are accessible, since only $m(\leq n)$ states of the plant are accessible to be used for control of the system above. The factor by which the optimal cost (assuming all states are accessible) is increased is bounded above by [25]

$$\Delta J \leq \frac{\lambda_{max}\{S(k_0)\}}{\lambda_{min}\{P(k_0)\}}$$

where $S(k)$ is the solution of the matrix difference equation

$$S(k) = G(k)^T S(k+1)G(k) + \hat{L}(k)^T A(k)^T P(k+1)B(k)[R(k)+B(k)^T P(k+1)B(k)]^{-1} \cdot B(k)^T P(k+1)A(k)\hat{L}(k) \quad (4.6)$$

and $S(N) = 0$.

For the application to be considered in this research, the procedure given by Dellon has several restrictions which must be removed, namely:

1. no provision is made for terminal cost;
2. the computational procedure used to compute the matrix $H(k)$ restricts $(n-m)$ to be less than or equal to 3;
3. the details of the method are not given for a multirate system.

Modifications to Dellon's Method

The following modifications were made to remove the restrictions noted above.

A provision was made to handle a terminal cost by generalizing the cost functional to

$$J(u(k),x(k_0),k_0) = \frac{1}{2} \{x(N)^T Q' x(N) + \sum_{i=k_0}^{N-1} [y(i)^T Q(i) y(i) + u(i)^T R(i) u(i)]\}.$$

Accordingly, the terminal condition of the Riccati equation (4.3) is changed to

$$P(N) = Q'.$$

Dellon's thesis [25] provides a computational procedure to find the H(k) matrix for values of $(n-m) \leq 3$. It was necessary to solve for H(k) when $(n-m) = 5$ (see Chapter 5). In order to obtain a tractible computational procedure which could be implemented as a computer program, it was necessary to find a more general method than the method used by Dellon. The method described below was developed.

Solving (4.2) for an H(k) matrix, it is seen that all the coefficients of the characteristic equation of the G(k) matrix must equal zero since all eigenvalues of G(k) equal zero. This is expressed in the following equation.

$$\det\{\lambda I - G(k)\} = \det\{\lambda I - A_{22}(k) - H(k)A_{12}(k)\}$$

$$= \lambda^{nm} + 0 \cdot \lambda^{nm-1} + 0 \cdot \lambda^{nm-2} + \cdots 0 \cdot \lambda + 0. \qquad (4.7)$$

The coefficients of the characteristic equation may be easily obtained as a function of the elements of the H(k) matrix as shown below [34]

$$\alpha_{nm-i} = - \text{Trace } [- A_{22}(k) - H(k)\ A_{12}(k)]^i = 0$$

where α_j is the coefficient of λ^j in (4.7). Thus, a set of nonlinear algebraic equations in the elements of H(k) must be solved for H(k) as shown below.

$$\alpha_i = 0 \qquad \text{for } i = 0, 1, \cdots, (nm-1)$$

This set of equations has a solution if $\{A_{22}(k), A_{12}(k)\}$ is an observable pair for $k \epsilon [k_o, N-1]$, [33]. This is true if the plant is uniformly completely observable [25], as originally assumed. The equations may be solved in a straightforward manner using a gradient search method, since the gradient can be easily computed (by the computer program) from the equation

$$\frac{\partial \alpha_{nm-i}}{\partial h_{ij}} = i \text{ Trace } \left\{[A_{22}(k) + H(k)A_{12}(k)]^{i-1} \frac{\partial H(k)}{\partial h_{ij}} A_{12}(k)\right\}$$

where h_{jk} is an element of $H(k)$.

Straightforward application of Dellon's technique to a plant whose output is multirate sampled results in a $C_1(k)$ matrix which is time varying and singular for certain values of k. The last modification to Dellon's technique involves changes in the problem formulation which must be made to insure that $C_1(k)$ is nonsingular for all k. It will be shown that the multirate sampling of the plant output may be described by difference equations in which $C_1(k)$ is a constant nonsingular matrix, if the plant transition matrix is expanded. Further, it will be shown that this expanded system representation (referred to as system A below) has the same cost as a simpler single rate system (referred to as system B below).

Although this modification applies to general multirate systems of the class considered, it will be described using the following simple system

$$x(k+1) = A_{3x3}x(k) + B_{3x1}u(k), \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \qquad (4.8)$$

Assume that the third order linear time invariant plant has its first state, $x_1(k)$, output every sample, and its second state output every other sample. This third state is inaccessible. Thus, the output equation and $C_1(k)$ are given by

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k), \qquad C(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for k an even integer}$$

and

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(k), \qquad C(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for k an odd integer.}$$

It is seen that $C_1^{-1}(k)$ does not exist for k an odd integer. Since Dellon's procedure requires that $C_1^{-1}(k)$ exist for all k, a modification must be made. The required modification can be accomplished by adding an extra state to the plant so that $x_2(k)$ is output only for k an even integer. For example, (4.8) becomes

$$x'(k+1) = \begin{bmatrix} a_{11} & 0 & a_{12} & a_{13} \\ 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{32} & a_{33} \end{bmatrix} \underbrace{\begin{bmatrix} x_1(k) \\ x_e(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}}_{\triangleq\, x'(k)} + \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ b_3 \end{bmatrix} u(k) \qquad (4.9)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x'(k), \qquad C(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for k an even integer}$$

$$x'(k + 1) = \begin{bmatrix} a_{11} & 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{22} & a_{23} \\ a_{21} & 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{32} & a_{33} \end{bmatrix} x'(k) + \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ b_3 \end{bmatrix} u(k) \qquad (4.10)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x'(k), \quad C(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } k \text{ an odd integer.}$$

The second plant state, x_e, is an artificial state used only for output purposes. It should be noted that the initial values of x_e and x_2 are identical. The system described by (4.9) and (4.10), referred to as system A, is physically identical to the original system.

In general it is seen that if the plant has n states of which f are sampled at the fastest rate and s are sampled at slower rates, then for design purposes the plant must have (n + s) states and the controller must have (n-f) states since

$$nm = (n+s) - (s+f) = n-f.$$

However, it will be shown below that the s extra plant states and the s slow sampled inputs to the controller may be neglected for design purposes without affecting the resulting design cost.

Consider a system of the form shown below (system B)

$$\begin{aligned} x(k+1) &= A_{3x3}x(k) + B_{3x1}\ u(k) \\ y(k) &= [1 \quad 0 \quad 0]\ x(k) \end{aligned} \qquad (4.11)$$

where the A and B matrices are the same as those of (4.8). This is a single rate system with only the first plant state being output. Since it is assumed that (4.8) is uniformly completely observable, then (4.11) is completely observable, since the first plant state is output at the fastest rate in both systems. Thus, a solution for H(k) in (4.2) exists and is of the form shown below.

$$G(k) = \left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} [\ a_{12} \quad a_{13}\ ] \right) \tag{4.12}$$

It is seen that the G(k) matrix for system A can be exactly the same as the G(k) in (4.12) by simply augmenting the H(k) matrix with a column of zeros. Thus G(k) becomes

$$G(k) = \left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix} \begin{bmatrix} a_{12} & a_{13} \\ 0 & 0 \end{bmatrix} \right) \quad \text{for k an even integer}$$

$$G(k) = \left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix} \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \right) \quad \text{for k an odd integer.} \tag{4.13}$$

Further, consider another similarity of systems A and B. By examining A(k) and B(k) in (4.9) and (4.10) in conjunction with (4.3) and (4.4), it is seen that

$$P(k) = \begin{bmatrix} P_{11} & 0 & P_{12} & P_{13} \\ 0 & 0 & 0 & 0 \\ P_{21} & 0 & P_{22} & P_{23} \\ P_{31} & 0 & P_{32} & P_{33} \end{bmatrix} \quad \text{and } K(k) = [k_1 \quad 0 \quad k_2 \quad k_3] \quad \text{for all k} \tag{4.14}$$

where $P(k) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$ and $K(k) = [k_1 \; k_2 \; k_3]$ (4.15)

is the solution of the Riccati equation for system B, assuming that there is zero cost associated with the extra plant state, x_e, in (4.9) and (4.10).

It can now be shown that the cost of system A is the same as the cost of system B.

The cost incurred in using Dellon's controller is given by [25]

$$J(x(k_0),k_0) = \frac{1}{2}\, x(k_0)^T\, P(k_0)x(k_0) + \frac{1}{2}\, \eta(k_0)^T\, S(k_0)\eta(k_0) \qquad (4.16)$$

where η is the solution to the difference equation

$$\eta(k+1) = G(k)\eta(k) \qquad (4.17)$$

and $\eta(k_0) = [H(k_0) \quad I]\; x(k_0) - H(k_0)C_1^{-1}(k_0)y(k_0)$ (4.18)

Writing (4.18) for system A gives

$$\eta(k_0) = \begin{bmatrix} h_1 & 0 & 1 & 0 \\ h_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k_0) \\ x_e(k_0) \\ x_2(k_0) \\ x_3(k_0) \end{bmatrix} - \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(k_0) \\ x_e(k_0) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ x_3(k_0) \end{bmatrix} \qquad (4.19)$$

where x_e is the extra state added to the plant. For system B, $\eta(k_o)$ is

$$\eta(k_0) = \begin{bmatrix} h_1 & 1 & 0 \\ h_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \\ x_3(k_0) \end{bmatrix} - \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} s_1(k_0) = \begin{bmatrix} x_2(k_0) \\ x_3(k_0) \end{bmatrix} \qquad (4.20)$$

It is seen that $\eta(k_0)$ will be the same for both systems. Further, it can be seen that $S(k_0)$ is the same for both systems by considering (4.6), (4.12), (4.13), (4.14), and (4.15). Therefore, the cost of both systems given by (4.16) is the same.

It should be noted that for system A, $x_e(k_0)$ is known and is equal to $x_2(k_0)$. Thus, the first state of the controller may be initialized to reduce the first state of the error vector, $\eta(k_0)$, to zero. That is, $\eta(k_0)$ becomes

$$\eta(k_0) = \begin{bmatrix} 0 \\ x_3(k_0) \end{bmatrix}$$

It is assumed that system B is similarly initialized.

There are three advantages to using the simplified system such as (4.11) for design purposes: (1) the computation of the H(k) matrix is simplified; (2) the order of the Riccati equation and other matrices is reduced; (3) the upper bound on the cost degradation is finite. It is easily seen that $\lambda_{min}\{P(k_0)\}$ for $P\{k_0\}$ given in (4.14) is zero. Thus, ΔJ given by (4.5) is infinite. This estimate of the cost degradation is unrealistic and corresponds to the plant initial condition

$$x_1(k_0) = x_2(k_0) = x_3(k_0) = 0, \quad x_e(k_0) \neq 0$$

where x_e is the extra plant state added in (4.9). However, this initial condition never occurs since $x_e(k_0) = x_2(k_0)$ for system A. When using the simplified plant equations, such as (4.11), the zero eigenvalue of $P(k_0)$ is removed.

The design procedure outlined in the last two sections was implemented as a computer program along with a procedure which simulates the resulting system. The simulation provides information about the dynamic response of the designed system's states to various initial conditions, and also provides a method of checking the cost of the designed system for specific initial conditions.

Specific Optimal Control Technique

A specific optimal control technique which may be applied to linear time varying plants with inaccessible states is developed by Murtuza [30] for continuous systems. The structure of the controller is linear time varying with an arbitrary number of states to be specified by the design engineer. A time domain quadratic cost is assumed.

The design technique is discretized for this research. The method of obtaining the optimal controller parameters is described below.

Assume a system whose dynamics are described by the equations

$$x(k+1) = A(k)\ x(k) + B(k)u(k) \qquad \text{(plant)} \qquad (4.21)$$

$$u(k+1) = D(k)\ x(k) + \hat{C}(k)u(k) \qquad \text{(controller)} \qquad (4.22)$$

$$u(k_0) = C\ x(k_0) \qquad \text{(controller initialization)} \qquad (4.23)$$

with a cost functional of the form

$$J = \frac{1}{2} \sum_{k=k_0}^{N-1} [x(k)^T\ Q_1 x(k) + u(k)^T\ Ru(k)] + \frac{1}{2}\ x(N)^T\ Q_2 x(N).$$

The controller parameters are restricted to be zeros of the controller. Controller poles must be specified before beginning the optimization.

A composite system is formed, as shown below.

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A(k) & B(k) \\ D(k) & \hat{C}(k) \end{bmatrix}}_{\triangleq\, \hat{R}(k)} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

To obtain the cost, the Riccati equation

$$P(k) = \hat{R}(k)^T\ P(k+1)\ \hat{R}(k) + Q(k)$$

where $Q(k) = \begin{bmatrix} Q_1 & 0 \\ 0 & R \end{bmatrix}$ and $P(N) = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \end{bmatrix}$

must be solved. The cost is seen to equal

$$J = \frac{1}{2}\, x(k_0)^T\ \hat{P}(k_0)\ x(k_0)$$

where $$\hat{P}(k_0) = P_{11}C + C^T P_{21} + C^T P_{22} C + P_{12} C$$

and $$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right].$$

The relative cost of the specific optimal controller system is

$$J^* = \frac{x(k_0)^T\ \hat{P}(k_0)\ x(k_0)}{x(k_0)^T\ P^*(k_0)\ x(k_0)}$$

where P^* is the solution to the optimal Riccati equation using the plant of (4.21) assuming all states are accessible.

It is seen that

$$J^* = y^T D^T \hat{P}(k_0) D y$$

where $D^T P^* (k_0) D = I$, and $y^T y = 1$.

Therefore, the maximum relative cost over all possible values of $x(k_0)$ is given by

$$\max_{x(k_0)} J^* = \lambda_{max}\{D^T \hat{P}(k_0) D\}$$

and the maximizing $x(k_0)$ is given by

$$x(k_0) = D y^*$$

where y^* is the eigenvector of $D^T \hat{P}(k_0) D$ which corresponds to λ_{max}. To find the gradient of the relative cost with respect to the controller parameters, the following difference equations are solved.

$$\frac{\partial P(k)}{\partial q_i} = 2 \frac{\partial R(k)^T}{\partial q_i} P(k+1) R(k) + R(k)^T \frac{\partial P(k+1)}{\partial q_i} R(k)$$

where $\dfrac{\partial J^*}{\partial q_i} = y^T D^T \dfrac{\partial P(k_0)}{\partial q_i} D y$, q_i is the i^{th} controller parameter,

and $\dfrac{\partial P(N)}{\partial q_i} = 0$.

The specific design procedure is outlined below:

1. find $P(k_0)$ for the initial parameters $\vec{q}$
2. find $x(k_0)$ such that the relative cost is maximum $(= J_1^*)$

3. solve for the gradient of J_1^* with respect to the controller parameters, $\vec{q}$, and step along the gradient until $J^* < J_1^*$. A Riccati equation must be solved for each step along the gradient.

4. go back to step 2 and iterate.

This design procedure was implemented as a computer program. The procedure described above was modified to provide the option of optimizing for a specific $x(k_0)$, rather than the worst case $x(k_0)$.

CHAPTER V

APPLICATION OF DESIGN PROCEDURES

The two existing design techniques described in Chapter IV and the new design technique developed in Chapters II and III are used to design a multirate cross-product steering controller for guidance of the Apollo spacecraft during midcourse correction burns [35]. The problems encountered in this design and the resulting controller designs are discussed in this chapter.

Statement of the Apollo Guidance Control Problem

During Apollo's short midcourse correction burns, it is necessary to control the basically unstable Apollo propulsion system by using information that is available from onboard measuring equipment -- heading and velocity of the spacecraft. This information is sampled at a fixed rate in order to insert it into an onboard digital computer which is programmed to produce signals to control the direction in which the rocket engine is pointed. The problem is to design a computer program which will: (1) stabilize the spacecraft guidance and propulsion system, (2) reduce any error in the increasing velocity of the spacecraft, (3) reduce the tumbling of the spacecraft at rocket shut off to near zero, and (4) require the rocket engine to be gimbaled only within fixed limits.

The dynamic behavior of the Apollo spacecraft (including the command module and LEM) moving on a typical trajectory can be adequately described by the vector difference equation, [21]

$$
\begin{bmatrix} v_a(K+1) \\ \theta_a(K+1) \\ q(K+1) \\ v_\perp(K+1) \\ w(K+1) \\ \delta(K+1) \end{bmatrix}
=
\begin{bmatrix}
1.0 & -.385 & -5. & 0 & 0 & -.165 \\
-.0065 & 1.0 & 0 & -.005 & .05 & 0 \\
.05 & 0 & 1.0 & .0385 & 0 & 0 \\
0 & .5 & 0 & 1.0 & 0 & -.5 \\
0 & 0 & .004075 & 0 & 1.0 & .0565 \\
0 & 0 & 0 & 0 & 0 & .5
\end{bmatrix}
\begin{bmatrix} v_a(K) \\ \theta_a(K) \\ q(K) \\ v_\perp(K) \\ w(K) \\ \delta(K) \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ .5 \end{bmatrix} \delta_c(K) \tag{5.1}
$$

using a sample period of 0.05 seconds. The velocity to be gained during the short midcourse burn determines the inertial reference direction and also the duration of the burn. The variables in this equation have the following significance:

v_a - 1.29 times the measured velocity at the inertial-measurement station excluding translational velocity in the inertial reference direction

θ_a - measured attitude at the inertial-measurement station relative to the inertial reference direction

q - generalized bending coordinate

$v_\perp$ - unwanted component of velocity of the center-of-mass of the vehicle. This component is perpendicular to the original velocity to be gained.

w - angular velocity of the underlying rigid body.

δ - angle of the nozzle of the main engine relative to the centerline of the bottom end of the service module.

δ_c - angle of the nozzle commanded by the guidance computer.

The relationship between the angles involved is shown in Figure 12.

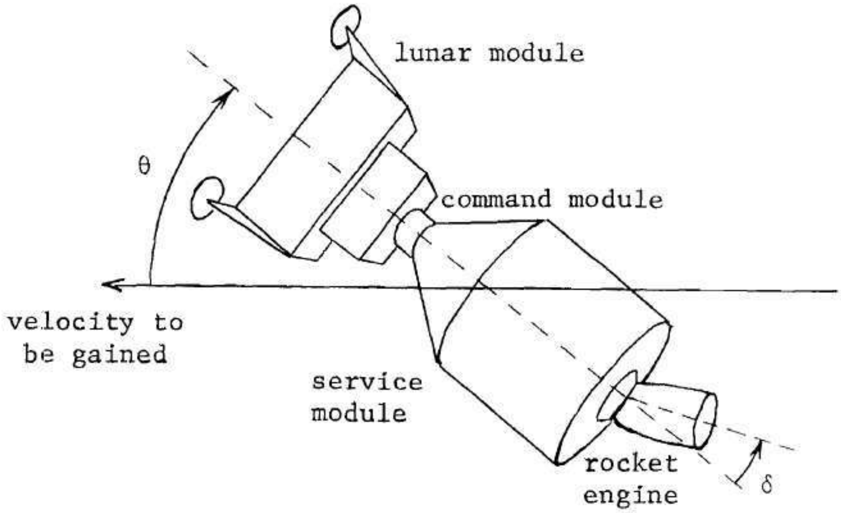


Figure 12. Definition of Angles Used to Describe Apollo Spacecraft Dynamics.

The object of the cross-product steering controller (guidance computer) is to minimize $|v_\perp|$ and $|w|$ at the termination of the burn, using the measurable plant outputs v_a and θ_a. Mechanical considerations typically constrain δ so that $|\delta| \leq 0.3$ radian. A penalty cost equal to the energy required to reduce $v_\perp$ and w to zero is used in the design cost functional

$$J = (185000)\ v_\perp^2(K_f) + (1100)\ w^2(K_f) + \frac{1}{2}\sum_{K=0}^{K_f-1} (100000)\ \delta^2(K) \qquad (5.2)$$

where K_f is the final sample of the burn. The last term of the cost functional is used to penalize large movements of the rocket nozzle. The weight of 100000 was used by Widnall and found to keep the nozzle angle within the required limits in his study [21].

Widnall studies a two loop multirate feedback configuration in which θ_a is sampled at the fast rate and v_a is sampled at a slower rate. This configuration is used as a starting point for the Apollo controller design.

Design Using Dellon's Method

The configuration of the optimal controller resulting from Dellon's design method is shown in Figure 13. Dellon's method dictated the choice of v_a for fast sampling so that the plant (which includes the samplers)

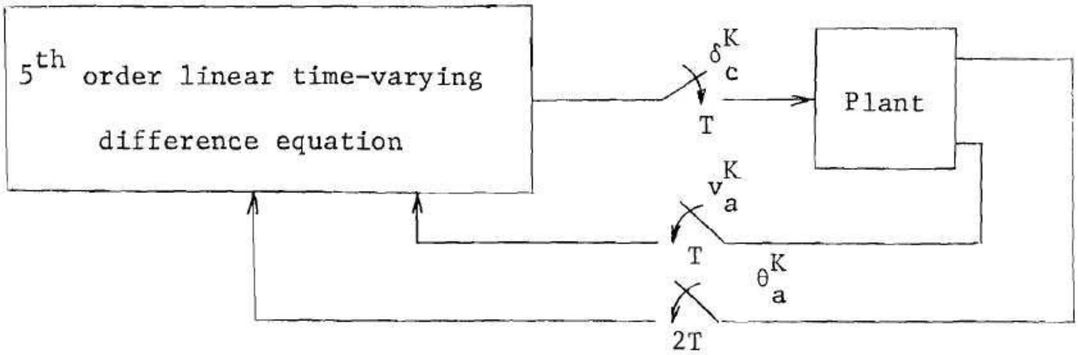


Figure 13. Structure of Apollo Guidance System Using Dellon's Method.

is uniformly completely observable, a requirement of the method. Further, it is characteristic of general optimal methods that the form of the controller cannot be specified a priori. Thus, a design with two uncoupled controllers, one for each loop, cannot be prescribed.

The form of the resulting optimal controller and the size of the required time varying controller matrices is shown below in Figure 14.

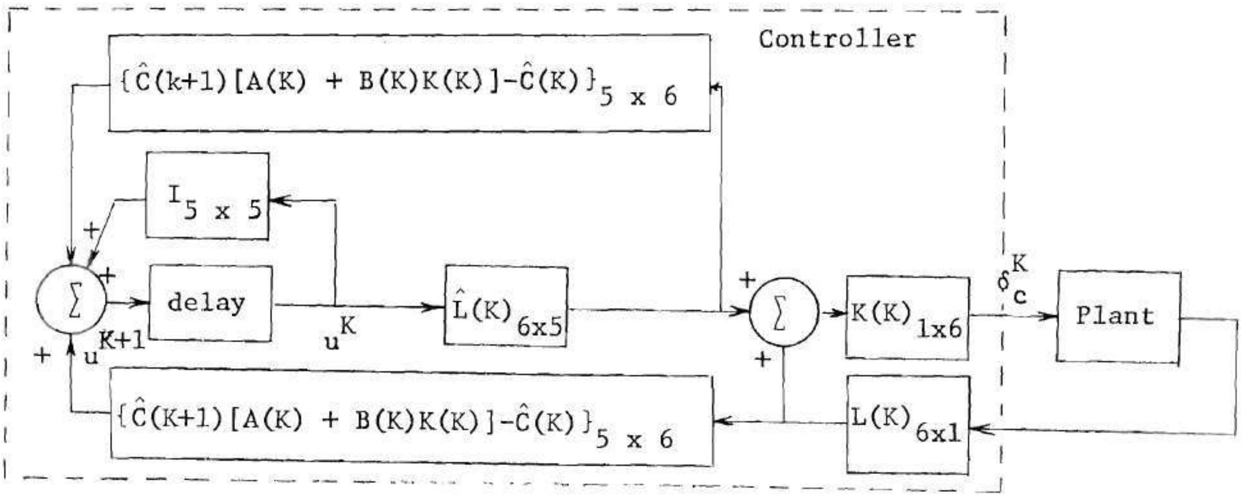


Figure 14. Matrices Used in Apollo Guidance Controller Resulting From Dellon's Method.

Preliminary Design Using the New Frequency Domain Design (FDD) Method

Use of the FDD technique makes it possible to specify a structure with two uncoupled controllers as shown in Figure 15. To facilitate comparison with the optimal method,

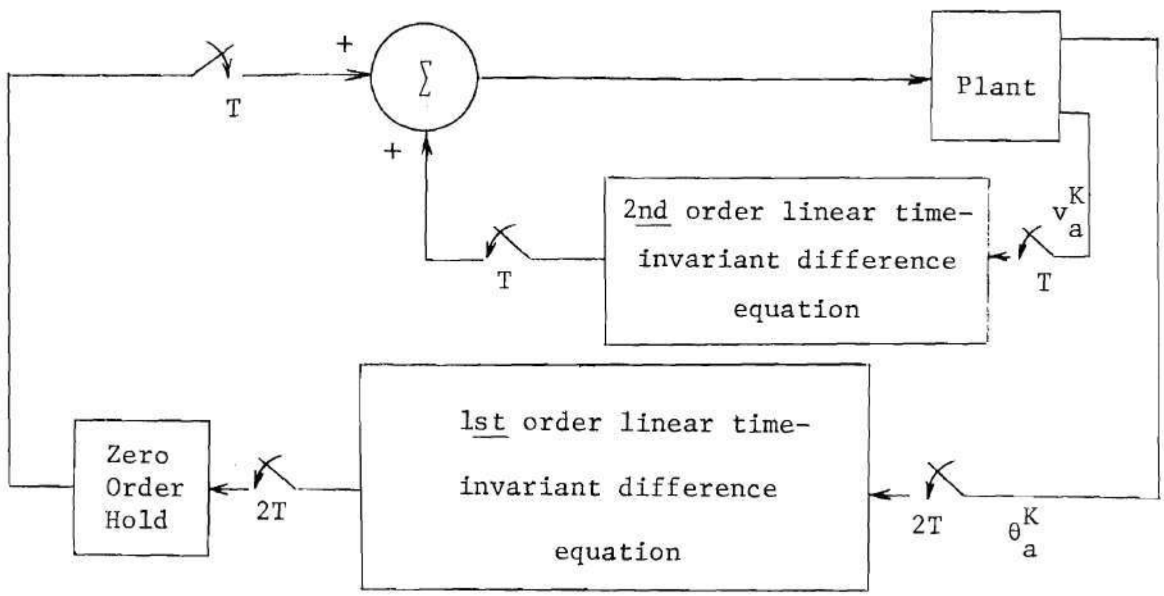


Figure 15. Structure of Apollo Guidance System Using the FDD Method.

the controller is structured to sample v_a at the fast rate. The choice of first and second order controllers as shown in Figure 15 follows from reasonable engineering judgement.

Preliminary Design Using the Specific Optimal Control (SOC) Method of Murtuza

Use of the SOC method allows specification of the controller structure by the design engineer, as is the case for the FDD method. For the same reasons as given in the preceeding section, the structure of the SOC controller is specified as shown in Figure 15.

Final Design of Apollo Guidance Controller Using SOC and FDD Methods

Certain problems encountered in the use of the SOC method required the use of the FDD method in conjunction with the SOC method. For this reason, the system designed using the SOC method is the same as the system resulting from the FDD design. The controller resulting from the combined FDD and SOC methods is shown in Figure 16, and is referred to as the "Hybrid Controller".

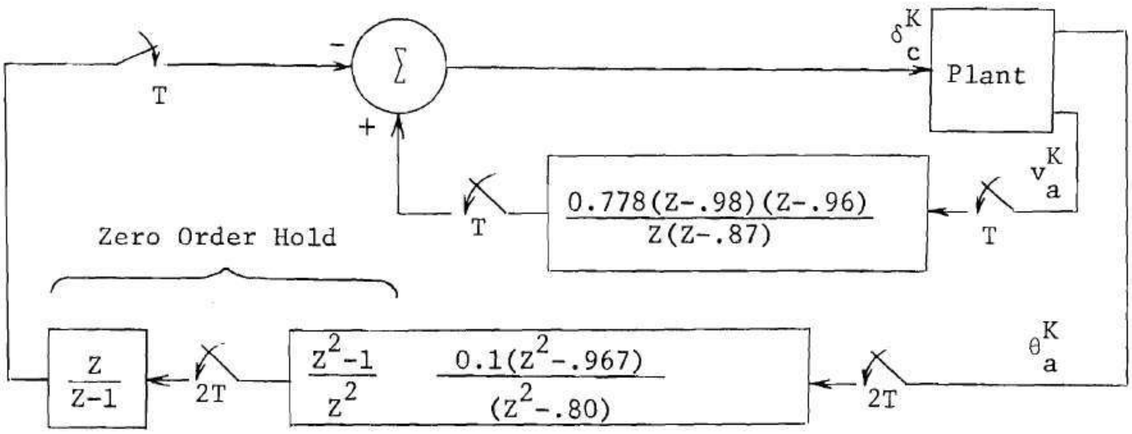


Figure 16. Hybrid Apollo Guidance Controller Resulting From the FDD and SOC Methods.

Implementation Using Each Method

Relatively few problems arise in the application of Dellon's method to the design of the controller. However, three aspects of this method, which require other than straightforward applications of the basic technique are discussed below.

As seen in Chapter IV, the plant must be completely observable when only the fast sampled outputs are used if the multirate plant is to be completely observable for every $k \epsilon [k_o, N-1]$. However, if θ_a is fast sampled, as suggested by Widnall, the plant is not observable for k an odd integer. For this reason v_a is used as the fast sampled output.

Numerical difficulties are sometimes encountered in solving for the H(k) matrix. For the Apollo problem a solution was obtained for plant sample periods, T of 0.1 and 0.05 seconds. However, the computerized computation procedure did not successfully solve for H(k) when T = 0.03 and 0.025 second. The norm of the gradient of α, mentioned in Chapter IV, approached zero before a solution was obtained. It is believed that this problem is associated with the bunching of the plant poles about 1.0 in the z plane when the sample period is reduced.

The third aspect of Dellon's method which requires additional design work involves numerical transients in the controller. When the designed Apollo systems were simulated on the computer, it was found that a large transient occurred in the controller output signal. As mentioned earlier in this chapter, this variable, $\delta(k)$, is required to remain small: $|\delta(k)| \leq 0.3$ radian for all k. This is due to gimbaling restraints on the rocket engine. However, during the first 20 samples, δ reached values in excess of 100 radians. Increasing the cost weighting on $\delta(k)$ did not reduce these transients. Also, increasing the system sample rate had no effect on the transients. The transients are directly related to non-ideal initialization of the controller states. With ideal initialization, which requires knowledge of the inaccessible plant and is thus unrealizable, the cost of the system can be made the same as the optimal system in which all plant states are accessible. The increased cost of the inaccessible state controller is mainly due to the transients occurring in the first 20 samples of the Apollo burn.

A substantial amount of work is required when applying the discretized SOC method which involves finding controller parameters with which to start the optimization. Murtuza suggests using the zeros which result from the optimal steady state gains shown in the following controller equation

$$\delta_c(k) = k_1 v_a(k) + k_2 \theta_a(k) + k_3 q(k) + k_4 v\ (k) + k_5 w(k) + k_6 \delta(k)$$

where k_i are the optimal steady state gains. This equation can be rewritten in terms of the accessible states as

$$\delta_c(k) = k_1' v_a(k) + k_2' v_a(k+1) + k_3' v_a(k+2) + k_4' \theta_a(k) + k_5' \theta_a(k+1) + k_6' \theta_a(k+2). \tag{5.3}$$

This equation cannot be implemented as a controller since the variables at (k+1) and (k+2) are not available when $\delta_c(k)$ must be computed. The best implementable approximation to (5.3) is

$$\delta_c(k+2) = k_1' v_a(k) + k_2' v_a(k+1) + k_3' v_a(k+2) + k_4' \theta_a(k) + k_5' \theta_a(k+1) + k_6' \theta_a(k+2). \tag{5.4}$$

This equation has a 2T delay which is not present in the optimal system. For the Apollo problem, this delay causes the cost of the resulting system using (5.4) to exceed 10^{38}. Since this exceeds the UNIVAC 1108 capacity, on which this procedure is implemented, the optimization cannot be started. For continuous systems considered by Murtuza, this delay could be made arbitrarily small simply by placing the controller poles in the far left s plane. However for discrete systems, the corresponding controller pole location is the origin of the z plane as used in (5.4). This results in the minimum possible controller delay, which is still too large for the Apollo system.

The optimal controller parameters resulting from Dellon's design cannot be used because: (1) the optimal controller has more states than the specific optimal ocntroller, (2) the optimal controller is totally interacting, and (3) the optimal controller parameters are time varying.

The solution for the Apollo problem was to merge the SOC method with the FDD procedure. The FDD technique makes possible design in the frequency domain; thus the FDD technique provides insight into the placing of the initial controller parameters. While the FDD technique is capable of multirate design independent of other methods, the SOC design procedure can be used to refine the parameters resulting from a FDD procedure.

Root contours were used in conjunction with the FDD technique, and are described below for the Apollo system whose structure is shown in Figure 15.

The ideal location of the closed loop system poles is assumed to be the pole location of the optimal steady state closed loop system with all the states accessible. They are shown below in Figure 17.

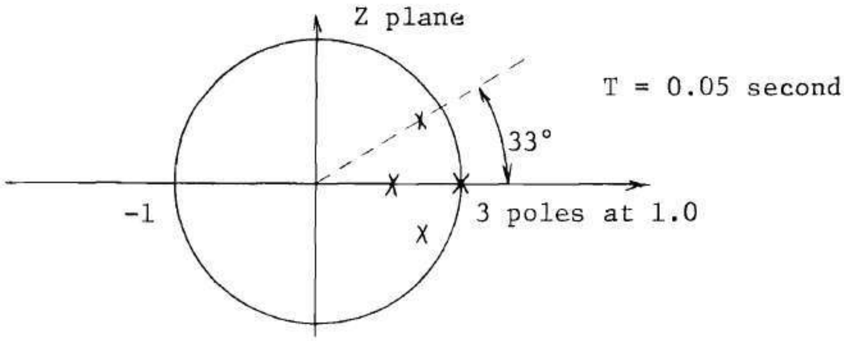


Figure 17. Optimal Pole Locations.

It is hoped that the closed loop poles can be put in this general arrangement with the root contours method. This is only a goal.

The root locus of the first (inner) loop is shown in Figure 18. The objective of this root locus is to stabilize the system by bringing

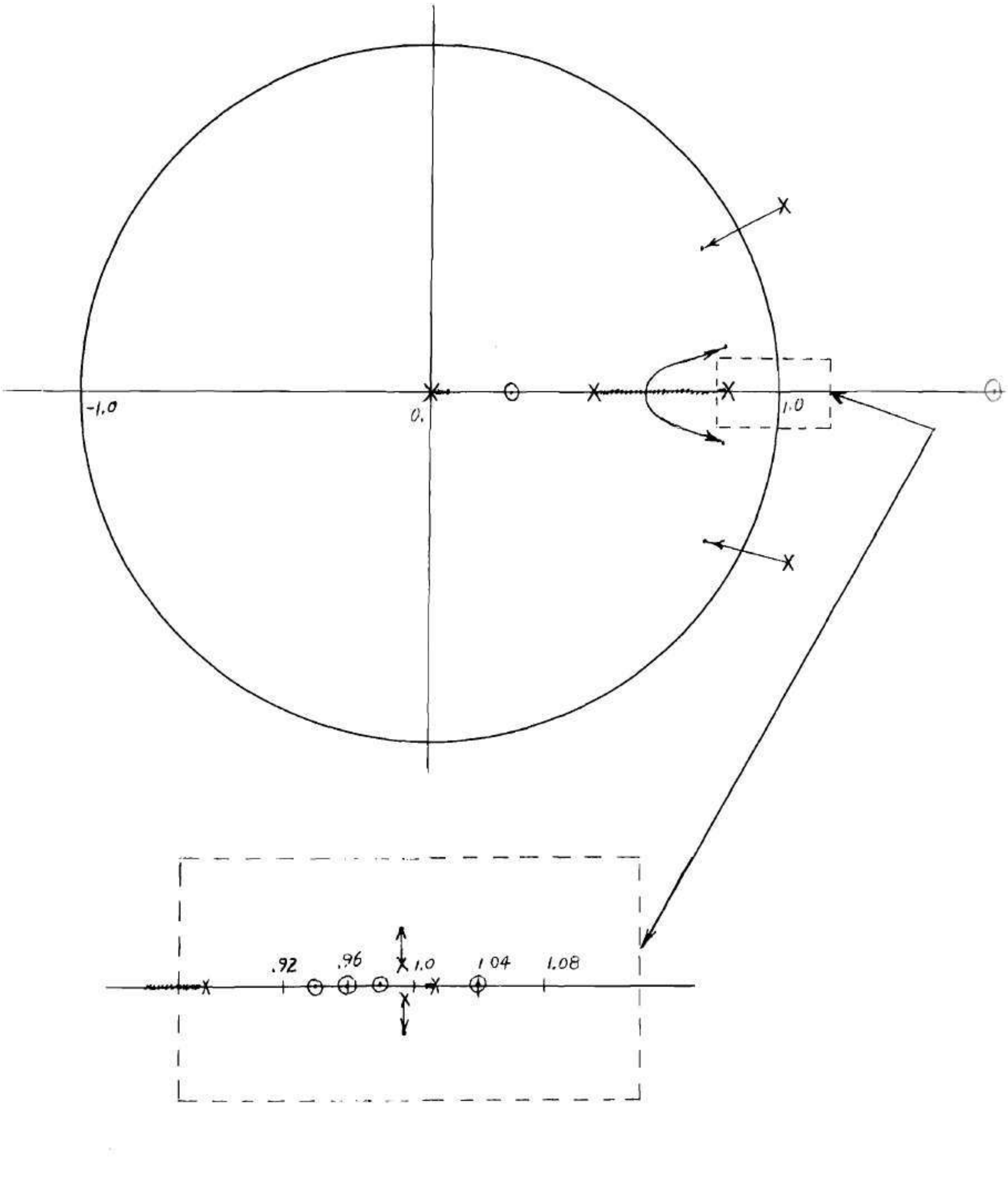


Figure 18. Root Locus for Inner Loop of System.

the two poles at $1.0 \pm j0.5$ within the unit circle. At the same time, deterioration of the stability due to the other poles had to be considered. The poles near 1.0 remained relatively fixed because of the nearby zeros.

The root contour for the second loop is impossible without using the FDD technique. The poles and zeros resulting from the first loop's design are shown in Figure 19.

The system open loop transfer function is transformed into the z^2 plane, as shown in Figure 20, and the root locus performed. The objective of this root locus is to move the poles near 1.0 inward to within the unit circle. The poles near 1.0 move rapidly since they are closely bunched. Thus, the gain to be used in the outer loop is small and the movement of the poles outside the area of interest near 1.0 is small.

The controller parameters resulting from the work above were used to start the SOC procedure's optimization. Although the controller parameters were changed only a small amount, the cost was reduced substantially.

Few problems were encountered in applying the FDD procedure. The two main problems were: (1) the FDD procedure becomes tedious for higher order systems, and (2) numerical inaccuracies occur when rapid pole movement occurs. This occurs when poles are bunched together. This problem is not particular to the FDD procedure, but occurs in any root locus study.

Discussion of Resulting Controller Designs

The complete Apollo guidance and propulsion systems are simulated

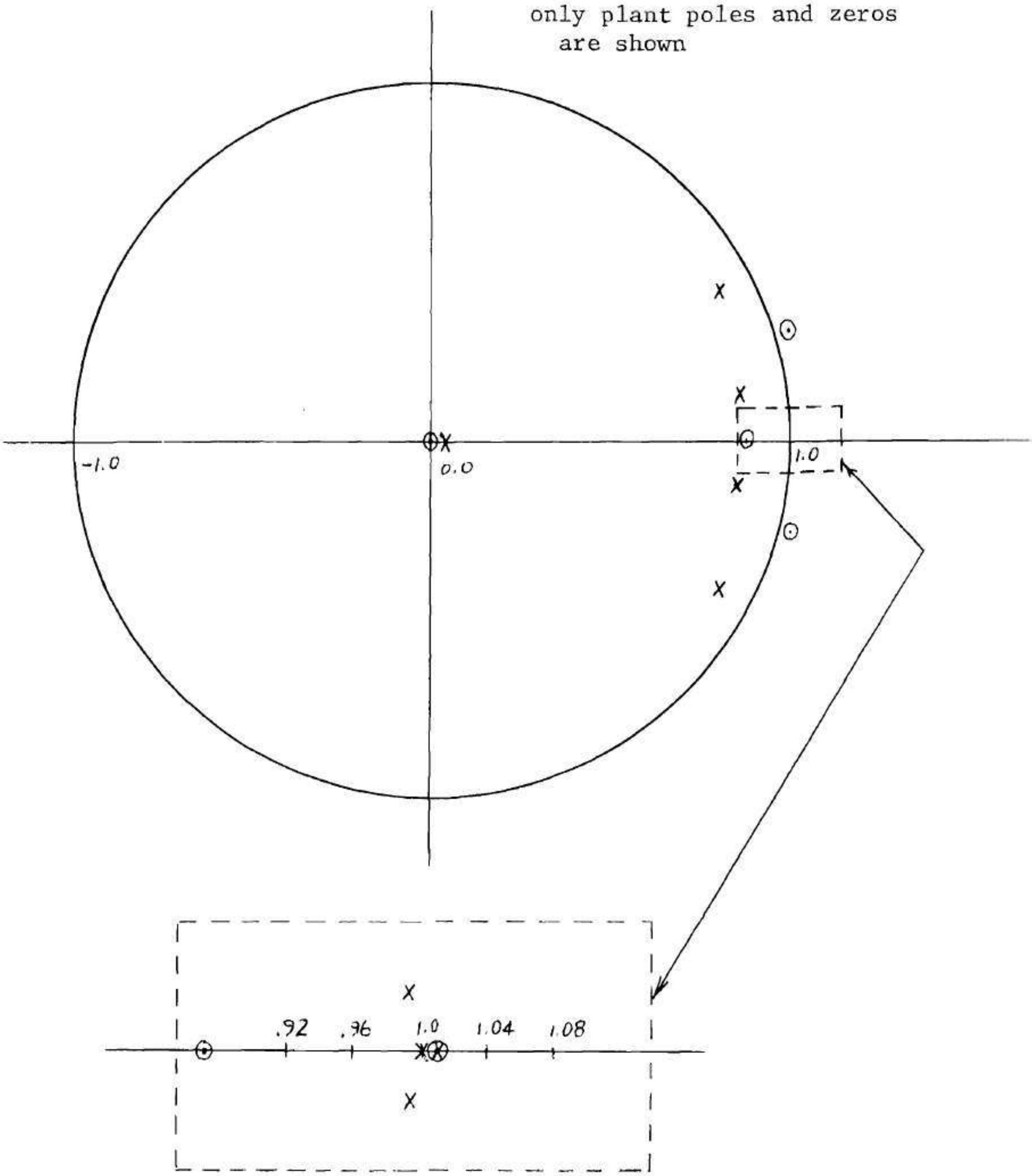
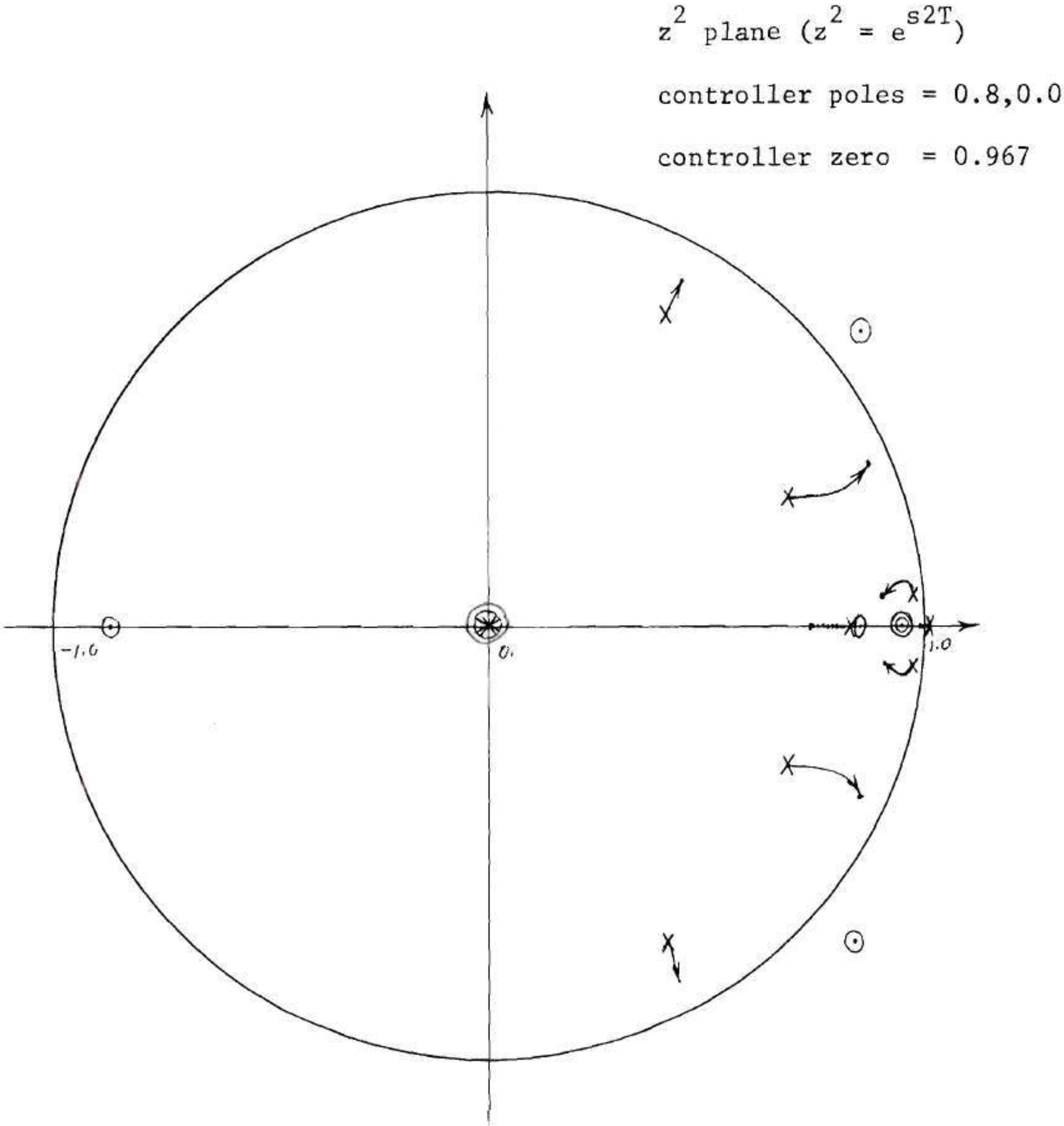


Figure 19. Open Loop Poles of Outer Loop of System.



Note: plant pole at 1.0 and controller zero at 1.0 are omitted.

Figure 20. Root Locus of Outer Loop of System.

using a UNIVAC 1108 computer. The costs and other important performance factors resulting from the simulation using each of the designed controllers are presented below.

For purposes of comparison, each of the designed systems is evaluated using the same plant initial conditions: all states are zero except $v_\perp(0) = 10.$ feet per second, and the corresponding value of the measured velocity $v_a(0) = -7.7$ feet per second; velocity to be gained is 100. feet per second. Although the perturbation in $v_\perp$ may occur at any time during the midcourse correction burn, the perturbation in $v_\perp$ at the initial time shows the essential system performance which occurs when $v_\perp$ is perturbed after the initial time.

The optimal controller is evaluated for two conditions: (1) the controller is correctly initialized assuming knowledge of the values of the inaccessible states of the plant, and (2) the controller is initialized without knowing the values of the inaccessible states. The resulting values are shown in Table 1. Condition (1) corresponds to a practical starting initial condition at $t = 0$; if $v_\perp$ is originally zero and becomes perturbed to a nonzero value during the burn, then condition (2) is more representative of the system's performance. The bound on δ is obviously violated for condition (2), and increasing the cost weighting on δ has little effect on this violation. This violation results from numerical transients within the controller's estimator during the first 20 samples.

Table 1. Cost and Performance Values of Guidance Control Systems

	Cost	$v_\perp(k_f)$	$w(k_f)$	max $\lvert\delta\rvert$
optimal controller				
condition (1) -	0.338×10^4	.0812 ft/sec	$-.011 \frac{rad}{sec}$	.024 rad
condition (2) -	0.337×10^{11}	1.81 ft/sec	$-.246 \frac{rad}{sec}$	252 rad
hybrid controller - (FDD and SOC)	0.730×10^7	1.29 ft/sec	$.029 \frac{rad}{sec}$	3.53 rad

The resulting performance of the hybrid design is shown in Table 1. Theoretically, the problem of correct initialization of the hybrid controller is not solved; for this reason the controller states are always initialized at zero. The hybrid controller violates the bound on δ, however not to the extent of the optimal controller of condition (2).

The sensitivity of the system cost and performance to bias errors in the measurement sensor of v_a was determined by adding a constant signal to the value of v_a input to the controller. A bias error of $\frac{1}{2}$ of one percent of the maximum v_a signal occurring in the previous simulations was added to the v_a signal. The performance of the hybrid and optimal systems was deteriorated as shown in Table 2.

Table 2. Cost and Performance Values with Bias Error

	Cost	$v_\perp(k_f)$	$w(k_f)$	max $\lvert\delta\rvert$
optimal controller				
condition (1) -	0.111×10^9	.0423 ft/sec	$-.011 \frac{rad}{sec}$	12.1 rad
condition (2) -	0.337×10^{11}	1.77 ft/sec	$-.246 \frac{rad}{sec}$	258. rad
hybrid controller -	0.745×10^7	-15.5 ft/sec	$-.001 \frac{rad}{sec}$	3.43 rad

It is seen that the optimal controller has a max $|\delta|$ of 12.1 radians as opposed to .024 radian with no bias error for conditions (1). Its cost is also increased from 0.338×10^4 to 0.111×10^9. The cost of the hybrid system is almost unchanged, however $v_{\perp}(k_f)$ is -15.5 ft/sec as opposed to 1.29 ft/sec for the unbiased system. The performance of the condition (2) optimal system is essentially unchanged.

The computer storage required of the hybrid controller is only 7 numbers. For the optimal controller, 36 parameters must be stored in the computer for each sample of the longest burn anticipated on the Apollo mission. This is in excess of 20 seconds; assuming a sample period of 0.05 seconds, 400 samples are needed, requiring a total storage capacity of 14,400 numbers, as shown in Table 3. The optimal controller also does more data processing than the hybrid controller, as the optimal controller has 5 states operating at the fast sample rate. The hybrid controller has only 3 states, one of which is operating at the slow sample rate. The respective members of additions and multiplications performed by each system per sample is shown in Table 3.

Table 3. Computer Storage and Computer Operations Needed

	Storage Capacity	Additions and Multiplications Per Sample	UNIVAC 1108 time needed to design controller
optimal controller	14,400 numbers	66 operations	41 seconds
hybrid controller	7 numbers	$14\frac{1}{2}$ operations	720^+ seconds

CHAPTER VI

CONCLUSIONS

A FDD procedure is developed which permits design of linear time invariant multirate multiloop sampled data control systems in the frequency domain. The use of the FDD technique and the use of the two existing design techniques are illustrated by applying all of these techniques to the design of guidance controllers for the Apollo spacecraft.

It is shown that the FDD technique is a useful method either in its own right as a complete tool for system design or in conjunction with Murtuza's modified specific optimal control design method. It is shown that the FDD technique is helpful in finding the initial controller parameters to be used in Murtuza's method, and Murtuza's method may be used to refine the parameters which are obtained by the FDD method alone. Further, the FDD technique makes use of the design engineer's experience in using classical design techniques on single rate sampled data systems to aid in designing multirate systems.

The existing design methods of Dellon and Murtuza require substantial modifications to allow their use on multirate systems. It is first necessary to discretize Murtuza's method and then to formulate a method of choosing the initial controller parameters. These modifications are described in Chapter IV and V. For Dellon's method, it is necessary to modify the procedure to either: (1) augment the plant equations to include the multirate plant output, or (2) use the simplification outlined in Chapter IV which makes possible the elimination of slow sampled plant outputs except for controller initialization. Further minor modifications of Dellon's method and Murtuza's method are described in Chapter IV.

A comparison of the three design methods considered in this research on the basis of general aspects of these methods and specific results of the Apollo design problem described in Chapter V is given in the following five subsections.

Structure of the Designed Controllers

The order of the controller, the computer storage required of the controller, and the computer operations needed per sample are compared below for the controllers resulting from each design method.

Order of the Controller

The optimal control design method of Dellon results in a controller with (n-f) states, where n is the order of the plant and f is the number of controller inputs which are sampled at the fastest rate. These controller states are totally interacting in general, and all operate at the fastest sample rate in the system so that a multirate system does not in fact result.

The SOC design method of Murtuza and the FDD procedure allow the order of the controller to be specified by the design engineer. The controller states may or may not be interacting and any number of the controller states may operate at sample rates slower than the fastest sample rate in the system subject to the restrictions given in Chapter 2.

Computer Storage Needed by the Controller

Dellon's method results in a controller that requires considerable computer storage capacity because of the time varying nature of the controller parameters, and due to the total interaction of the controller states; for example, in the Apollo problem 14,400 numbers must be stored (see Table 3). Murtuza's method and the FDD procedure require minimal storage since the resulting controller's parameters are time invariant, and since the controller states are not totally interacting in general. For these methods a storage of only 7 numbers is required in the Apollo problem.

Computer Operations Per Sample of the Controller

Dellon's method results in a controller which performs considerably more additions and multiplications per sample (fastest sample rate) than a corresponding controller resulting from Murtuza's method or the FDD procedure with approximately the same number of states. This is because Dellon's controller is totally interacting and operates at the fastest sample rate. In general, this is not the case for the other two controllers.

The number of computer operations per sample is roughly five times as large for the optimal controller as for the hybrid controller based on the Apollo problem (see Table 3).

Design Time

The amounts of computer design time and engineering design time required by the three different design methods are discussed below.

Computer Design Time

Based on the Apollo problem discussed in Chapter V, the computer design time needed for the SOC design method of Murtuza is at least twenty times as great as the computer design time needed for Dellon's method.

Computer design time will always be greater for Murtuza's method than for Dellon's method. Only one matrix difference equation must be solved one time for Dellon's method while several larger matrix difference equations must be solved a number of times for Murtuza's method.

The FDD procedure does not require computer use. However, a computerized root locus procedure was used in working the Apollo problem. This procedure required a small amount of computer time in comparison to Dellon's method.

Engineering Design Time

Dellon's method requires relatively little engineering manipulation once this method has been implemented as a computer program. Determination of the cost functional weighting factors may require some engineering skill to obtain desired system time responses.

Murtuza's method requires more engineering manipulation than Dellon's method. The following problems must be solved using engineering judgement: (1) determination of cost functional weighting factors, (2) determination of the controller structure, and (3) determination of the initial controller parameters. Two problems are associated with (3) above. First, the initial controller parameters must result in a system for which the cost is less than the maximum allowable number of the computer used for this design. Second, the initial controller parameters must allow the parameter search procedure used by Murtuza to find a local minimum of the cost functional which is tolerably close to the optimal cost. For high order systems, this is often a trial and error procedure. If the initial controller parameters are to be based on the optimal steady state gains, then the sample period of the system must be reduced for some systems in order to reduce the delay resulting from the discrete controller. The increased system sample rate will in turn require increased computer design time.

The FDD procedure requires considerable engineering manipulation, and skill in using the classical design techniques for single rate sampled data systems.

Controller Transients

One of the major problems arising in the Apollo problem is controller transients occurring within the first 20 samples. These transients are discussed below for the systems designed by the three different methods.

The cost of the system resulting from Dellon's design method is given by

$$J(x(k_o),k_o) = \frac{1}{2} x^T(k_o)\ P(k_o)\ x(k_o) + \frac{1}{2} \eta^T(k_o)\ S(k_o)\ \eta(k_o) \tag{6.1}$$

where $$\eta(k+1) = G(k)\ \eta(k) \tag{6.2}$$

$\eta(k) = \hat{C}(k)\ x(k) - z(k)$ = error of controller states

$z(k)$ are the controller states (see Chapter IV for full explanation of the expressions shown above).

Since the cost of the optimal system in which all plant states are accessible is $\frac{1}{2} X^T(k_o)\, P(k_o)\, x(k_o)$, it is seen that the degradation in this optimal cost is proportional to the term $\eta^T(k_o)\, S(k_o)\, \eta(k_o)$. If the controller states are initialized as proposed by Dellon

$$z(k_o) = H(k_o)\, C_1^{-1}(k_o)\, y(k_o)$$

then

$$\eta(k_o) = \begin{bmatrix} x_{m+1}(k_o) \\ x_{m+2}(k_o) \\ \cdot \\ \cdot \\ \cdot \\ x_n(k_o) \end{bmatrix}$$

where $x_i(k_o)$, $i = (m+1), \cdots, n$ are the inaccessible plant states. Since $G(k)$ in (6.2) has all zero eigenvalues, the norm of $\eta(k)$ becomes zero after a few samples. Thus, all cost degradation results in the first few samples. It is also seen that if the inaccessible plant states are available at k_o, then the controller could be ideally initialized resulting in no cost degradation. Thus, the cost degradation for the system resulting from Dellon's method is due to the controller transients occurring in the first few samples resulting from nonideal controller initialization.

Further, it is seen that these transients will be most severe when the ratio

$$\frac{\lambda_{max}\{S(k_o)\}}{\lambda_{min}\{P(k_o)\}}$$

is large. This is the case for the Apollo problem described in Chapter V. For that problem the ratio above was equal to 0.5×10^{15}.

Based on the Apollo problem, the transients of the controllers, resulting from Murtuza's method and the FDD method, are not as severe as those observed in the optimal controller for nonideal initialization resulting from Dellon's method (see Table 1 in Chapter V). As a result, the optimal cost is not degraded for these systems to the extent of the system resulting from Dellon's design method.

Bias Error

It is seen that the performance of all the Apollo guidance systems described in Chapter V is seriously deteriorated by inserting a small bias error into one of the measurement signals input to the controller (see Table 2 in Chapter V). This deterioration in performance is in part due to the lack of a pure integrator in the feedback path of any of these systems, which will reduce the steady state error of $v_{\perp}$ to zero. The addition of such an integrator is straight forward to carry out in the FDD procedure. The integrator may be added to the optimal system by artificial means, such as inserting an extra state in the plant equations before beginning the optimal design.

Multirate Sampling

The advantages of multirate sampling are essentially lost when using Dellon's design method since all the controller states must operate at the fastest sample rate. Slow sampled controller inputs are used only for controller initialization, as shown in Chapter IV.

The SOC design method of Murtuza, and the FDD method use multirate sampling to reduce the number of operations per sample necessary for the controller to carry out. This is done by allowing different controller states to operate at different sample rates. In the Apollo problem the use of multirate sampling reduced the required computer storage from 14,400 to seven numbers, and reduced the required computer operations per sample from 66 to $14\frac{1}{2}$ (see Table 3).

APPENDIX

PROPERTY OF A POLYNOMIAL OF A COMPLEX VARIABLE

It is shown below that sums of polynomials of the form considered in Chapter II may be simplifed considerably. Consider a polynomial of the form of (2.8) shown below.

$$P(z) = \sum_{k=0}^{p-1} (z + a_1 e^{\frac{k}{p}})(z + a_2 e^{\frac{k}{p}}) \cdots (z + a_n e^{\frac{k}{p}})\, e^{\frac{km}{p}} \qquad \text{(A.1)}$$

where $e^x \triangleq \exp(j2\pi x)$, $m = \rho-\zeta$, $n = p\rho-m$, and a_i are complex numbers. These constants are described below:

ρ = number of poles in each term of sums such as term A of (2.2)

ζ = number of zeros in each term of sums such as term A of (2.2)

a_i complex numbers similar to the a_i of (2.5)

m, ρ, and ζ are non-negative integers

p a positive integer related to the ratio of the sample period of involved transfer functions (e.g., p = 4 in (2.5)).

Some terms of the sum shown above are written in detail in Figure 21.

$$
\begin{array}{lllll}
 & \overbrace{}^{S=0\ \text{column}} & \overbrace{\phantom{z^{n-1}(a_1 + a_2 + \cdots + a_n)\, e^0}}^{S=1\ \text{column}} & & \overbrace{}^{S=n\ \text{column}} \\
P(z) = & z^n e^0 & + \; z^{n-1}(a_1 + a_2 + \cdots + a_n)\, e^0 & + \cdots + & (a_1 a_2 \cdots a_n)\, e^0 \quad + \\
 & + \; z^n e^{\frac{m}{p}} & + \; z^{n-1}(a_1 + a_2 + \cdots + a_n)\, e^{\frac{m+1}{p}} & + \cdots + & (a_1 a_2 \cdots a_n)\, e^{\frac{m+n}{p}} \quad + \\
 & + \; z^n e^{\frac{2m}{p}} & + \; z^{n-1}(a_1 + a_2 + \cdots + a_n)\, e^{\frac{2(m+1)}{p}} & + \cdots + & (a_1 a_2 \cdots a_n)\, e^{\frac{2(m+n)}{p}} \quad + \\
 & \vdots & & \vdots & \\
 & + \; z^n e^{(p-1)\frac{m}{p}} & + \; z^{n-1}(a_1 + a_2 + \cdots + a_n)\, e^{\frac{(p-1)(m+1)}{p}} & + \cdots + & (a_1 a_2 \cdots a_n)\, e^{\frac{(p-1)(m+n)}{p}}
\end{array}
$$

Figure 21. Expansion of Polynomial P(z).

Notice all the bracketed terms in each column shown in Figure 21 are the same. Some of these columns add to zero because of the phase shift e^x involved in the separate terms. It is known that

$$\sum_{q=0}^{p-1} e^{\frac{q}{p}} = 0 \quad \text{and} \quad \sum_{q=0}^{p-1} e^{\frac{q(m+s)}{p}} \begin{cases} = 0 & \text{for } (m+s) \neq 0, p, 2p, \cdots \\ = p & \text{for } (m+s) = 0, p, 2p, \cdots \end{cases}$$

Thus, the only columns in Figure 21 which add to a nonzero sum are the columns in which

$$s = -m,\ p-m,\ 2p-m, \tag{A.2}$$

where it is understood that $s \geq 0$. Let $(\eta p-m)$ be the smallest non-negative s in the sequence (A.2); that is η is the smallest integer such that $\eta \geq \frac{m}{p}$. Thus, the highest degree of z to appear in P(z) is

$$n - \eta p+m = p\rho-m-\eta p+m = p(\rho-\eta)$$

Therefore P(z) may be written as below

$$P(z) = p\ \{\ \alpha_{n-p(\rho-\eta)} z^{p(\rho-\eta)} + \alpha_{n-p(\rho-\eta-1)} z^{p(\rho-\eta-1)} + \cdots + \alpha_{n-p} z^{p} + \alpha_{n}\ \}$$

where α_j are the sums of all products of all combinations of a_i values taken j at a time.

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